Extension and analysis of use of derivatives for compensation of hybrid solution of linear differential equations

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INTRODUCTION

When compared to continuous (analog) computation, hybrid computation is subject to two sources of error not associated with hardware, but caused by its logical nature. They are often referred to as the time (or transport) delay, and the reconstruction errors.

This time delay error is caused by the time taken for the digital computer to process the data sampled from the analog computer, before sending the updated results back to the analog. The reconstruction error results from the hold action of the digital-to-analog link: the updated value from the digital is sent to the analog and held fixed until the next updating, instead of being updated continuously.

The effect of these errors on the hybrid solution (as compared with a pure analog solution) is twofold. First, inaccuracies are introduced. Second, the hybrid solution may become instable and grow without bound, even though the correct solution is bounded or even decreases to zero.

To prevent instability and minimize error, hybrid computations utilize compensation techniques. The variables processed in the digital computer for use in the analog computer are calculated at some future time, by an extrapolation scheme, before being sent to the analog. Depending on the scheme used, this technique can have a beneficial effect on the accuracy and stability of the solution, for a given sampling interval.

There are a number of extrapolation techniques commonly used to achieve compensation. One such technique is that of multistep extrapolation, or digital filters, in which values of the variables at earlier time are used for extrapolation. A good discussion of this method is given by Mitchell.[1] He demonstrates its shortcomings for heavily damped systems, caused by the instability of the extraneous solutions introduced by use of values at earlier times. For each step back in time, one extraneous solution is introduced, and these solutions are instable for large enough sampling intervals. The popular three-step, or parabolic, extrapolation introduces two such solutions, and their amplitude increases with increasing damping, so that heavily damped systems require small sampling intervals for stability.

Some years ago, Miura and Iwata[2] suggested another technique of extrapolation. For solving differential equations, they used the derivative of each variable to extrapolate, rather in the manner of a Taylor series. The implementation suggested was to add to the output of an integrator a multiple of the input, the sum being the extrapolated value of the variable. Further use of this scheme, for undamped systems, was made by Gilbert[3] and Karplus[4,5] with several implementations suggested. Gilbert[3] analyzed the undamped system, using z-transforms. This extrapolation technique has the advantage of requiring either no backward steps, or only one, depending on the implementation, thus eliminating or reducing the number of extraneous solutions introduced. The result is a solution which is not only more accurate than the uncompensated hybrid solution, but can be more stable. This is in contrast to the use of multi-step methods, which improve the accuracy but reduce the stability compared to the uncompensated hybrid solution.

There is apparently only one published reference to the use of the method of Miura and Iwata for a damped second order system. Bekey and Karplus,[5] on pages 382-383 of Chapter 12, give some results of unpublished* work of Howe and Fogarty.[6] In this work, they extrapolate x and $\dot{x}$ by using 1.5 T times $\dot{x}$ and x respectively, where T is the sampling interval. They use an implementation where the extrapolation is performed in the analog computer, the extrapolated values are sampled by the digital computer, combined to give $\ddot{x}$, and then converted D to A and sent to the analog computer for integration. We can call this calculation of extrapolated values in the analog computer *analog* compensation. The analysis by z-transforms is based on a timing sequence in which the A to D sampling occurs *before* the D to A conversion of $\ddot{x}$. The result of this compensation scheme is two desirable solutions which have exponents whose error are of order $(\omega T)^2$, in contrast to error of order ωT for the uncompensated solution, where ω is the natural frequency. However, there are two extraneous solutions of the order $(\zeta\omega T)^{\frac{1}{2}}$, where ζ is the damping coefficient, in contrast to the single extraneous solution of order $\zeta\omega T$ for no compensation. Therefore, we see that in this case derivative compensation improves the accuracy, but it reduces the stability, compared to no compensation.

This situation can be improved if we change to what might be called *digital* compensation. Here, we sample x and $\dot{x}$, and do the extrapolations in the digital computer. This is the scheme used in the present report. For a damped system, it uses no backward time steps, instead of the one backward step inherent in the Howe-Fogarty implementation. Therefore, it has only one extraneous solution, of order $\zeta\omega T$, and is somewhat more stable than the uncompensated case because of a better numerical factor. The accuracy of the two desirable solutions is of the same order as those of Howe and Fogarty.

The same scheme as that given for digital compensation in this report can be obtained by the analog compensation method of Howe and Fogarty if they change the order of A to D sampling and D to A conversion and perform D to A *before* A to D. This may not be a desirable implementation because the transients set up by D to A may interfere with the values sampled A to D immediately thereafter.

The purpose of this report is to extend the use of derivatives for extrapolation, to apply the method to a damped second order system typical of control problems, to analyze the system by use of z-transforms, and to compare the analysis with hybrid calculations using both derivative compensation and multi-step compensation.

The extension of the derivative method, which is also referred to as Taylor series compensation, is in several directions. First, we not only correct x by using $\dot{x}$, but also by using $\ddot{x}$, since that derivative is also available. Second, we do not assume an extrapolation ahead by 1.5T, but carry along arbitrary constants which are then chosen to give greatest accuracy. The first order corrections are indeed found by this method to be 1.5T, providing a simple analytical derivation of this fact. The second order coefficient of $\ddot{x}$ may be chosen in several ways toenhance accuracy or stability.

The analysis is applied to a linear damped oscillator, forced by a control function which is a linear combination of x and $\dot{x}$. The oscillator is implemented on the analog computer, the control function on the digital computer.

The z-transform analysis yields formulas which can be used to predict the stability of both the compensated and uncompensated cases for any values of the parameters and sampling interval. Similar results are given for the three-step compensation scheme, and show it to be less stable.

A numerical test was made by implementing both schemes on a Beckman 2200/SDS 9300 hybrid computer. The hybrid calculations were compared with continuous calculations of the same system made on the analog computer. The superior accuracy and stability of the Taylor series method over the three-step method is clearly apparent in the strip chart results, as well as in the digital printouts.

Analysis

Continuous solution

The forced oscillator analyzed is defined by

$$x \times 2\omega\zeta\dot{x} + \omega^2 x = \omega^2\delta \tag{2.1}$$

$$\delta = Kx_c - K(\tau\dot{x} + x) \tag{2.2}$$

where K and τ are constant control parameters. The command input x_c is taken to be a constant here, for

* Prof. Fogarty kindly sent me a copy of this report, and the remarks in this paragraph are based on my analysis of Section 5 of the report.

ease of analysis. Further, only the simple initial conditions $x(0) = 0$, $\dot{x}(0) = 0$ are considered, although other values bring only algebraic complication.

The exact continuous solution of this problem is simply obtained by transposing the variables on the right side and defining total frequency and damping by

$$\omega^2_T = \omega^2(1 + K), \quad 2\omega_T \zeta_T = 2\omega\zeta + \omega^2 K\tau \quad (2.3)$$

The solution with zero initial conditions is then

$$x = \frac{Kx_c}{1 + K}\left[1 - \frac{1}{2}\left(1 - \frac{i\zeta_T}{\zeta_T^1}\right)e^{\lambda_{T1}t} - \frac{1}{2}\left(1 + \frac{i\zeta_T}{\zeta_T^1}\right)e^{\lambda_{T2}t}\right] \quad (2.4)$$

$$\zeta^1_T = (1 - \zeta^2_T)^{1/2}, \quad \lambda_{T1,2} = \omega_T(-\zeta^1_T \pm i\zeta_T^1)$$

where the $\lambda_{T1,2}$ are the roots of the characteristic equation

$$\lambda^2_T + 2\omega_T\zeta_T\lambda_T + \omega^2_T = \lambda^2_T + (2\omega\zeta + \omega^2 K\tau)\lambda_T + \omega^2(1 + K) = 0 \quad (2.5)$$

Hybrid difference Equations

The hybrid implementation considers the δ term as a control function which is calculated digitally while the left side of (2.1) is calculated continuously in the analog computer. Thus, between the sampling times nT and (n + 1)T, δ is held fixed at the value δ_{Pn} supplied to the analog at t = nT.

Therefore during this interval the analog solves

$$\ddot{x} + 2\omega\zeta\dot{x} + \omega^2 x = \omega^2\delta_{Pn} \quad (2.6a)$$

with initial conditions

$$x = x_n \qquad \dot{x} = \dot{x}_n \quad (2.6b)$$

The solution of (2.6) is

$$x = \left[\frac{\lambda_2(x_n - \delta_{Pn}) - \dot{x}_n}{\lambda_2 - \lambda_1}\right]e^{\lambda_1(t-nT)} + \left[\frac{\dot{x}_n - \lambda_1(x_n - \delta_{Pn})}{\lambda_2 - \lambda_1}\right]e^{\lambda_2(t-nT)} + \delta_{Pn}$$

$$\dot{x} = \lambda_1[\quad '' \quad]e^{\lambda_1(t-nT)} + \lambda_2[\quad '' \quad]e^{\lambda_2(t-nT)} \quad (2.7a, b)$$

where $\lambda_{1,2}$ are the roots of the free-vibration characteristic equation

$$\lambda^2 + 2\omega\zeta\lambda + \omega^2 = 0,$$
$$\lambda_{1,2} = \omega(-\zeta \pm i\zeta^1), \quad \zeta^1 = (1 - \zeta^2)^{1/2} \quad (2.8)$$

At t = (n + 1)T these are expressible in real form as

$$x_{n+1} = e^{-\omega\zeta T}[(x_n - \delta_{Pn})(\cos\omega\zeta^1 T + \zeta/\zeta^1 \sin\omega\zeta^1 T) + \dot{x}_n/\omega \sin\omega\zeta^1 T] + \delta_{Pn} \quad (2.9a)$$

$$\dot{x}_{n+1} = e^{-\omega\zeta T}[\dot{x}_n(\cos\omega\zeta^1 T - \zeta/\zeta^1 \sin\omega\zeta^1 T) - \omega(x_n - \delta_{Pn})\sin\omega\zeta^1 T] \quad (2.9b)$$

These two equations are difference relations between x_n, $\dot{x}_n$ and x_{n+1}, $\dot{x}_{n+1}$, with given δ_{Pn}. Equations (2.7) show that the analog computer produces segments of forced damped vibrations between sampling times, each joined to the adjacent segments with continuous x and $\dot{x}$, but discontinuous $\ddot{x}$, because δ_{Pn} changes at each sampling time. The hybrid system solves the difference equations (2.9), as will we, but first δ_{Pn} must be specified in terms of x and $\dot{x}$ to model the digital part of the calculation.

Taylor series compensation

The digital calculation of δ_{Pn}, the value sent to the analog at time nT, can only depend on quantities sampled by the digital at previous sampling times. We will project x and $\dot{x}$ and take δ_{Pn} to be given by the projected values according to (2.2):

$$\delta_{Pn} = Kx_c - K(\tau\dot{x}_{Pn} + x_{Pn}) \quad (2.10)$$

The projections are accomplished from x_{n-1}, $\dot{x}_{n-1}$ by a Taylor series form

$$x_{Pn} = x_{n-1} + \ell T\dot{x}_{n-1} + kT^2\ddot{x}_{n-1} \quad (2.11a)$$

$$\dot{x}_{Pn} = \dot{x}_{n-1} + hT\,\ddot{x}_{n-1} \quad (2.11b)$$

We have used as many terms as the available derivatives allow. The quantity $\dot{x}_{n-1}$ can be sampled and made available in the digital. The second derivative is calculated from the differential equation (2.6a)

$$\ddot{x}_{n-1} = -2\omega\zeta\,\dot{x}_{n-1} - \omega^2 x_{n-1} + \omega^2\delta_{P,n-1} \quad (2.12)$$

Equation (2.10)–(2.12) are the essence of the Taylor series compensation scheme proposed here. In contrast, a three-step scheme would project δ_{Pn} from previous δ's:

$$\delta_{Pn} = a_0 \delta_{n-1} + a_1 \delta_{n-2} + a_2 \delta_{n-3} \tag{2.13a}$$

where

$$\delta_{n-1} = Kx_c - K(\tau \dot{x}_{n-1} + x_{n-1}) \tag{2.13b}$$

and similarly for δ_{n-2}, δ_{n-3}. This scheme goes back to $(n - 3)T$, two steps further than (2.11).

In both cases the constants ℓ, k, h, or a_0, a_1, a_2 are available to help improve the solution. For the three-step method, it is conventional to project to the time $(n + 1/2)T$, for which the values of the constants are

$$a_1 = -21/4, \; a_2 = 15/8, \; a_0 = 1 - a_1 - a_2 = 35/8 \tag{2.14}$$

If we project (2.11) the same distance, we find

$$\ell = h = 3/2, \qquad k = 9/8 \tag{2.15}$$

Instead we will carry the constants along, and choose their values on the basis of the resulting formulas.

The final form of δ_{Pn} comes by inserting (2.11) and (2.12) into (2.10) to obtain

$$\begin{aligned}\delta_{Pn} = Kx_c - K\{&x_{n-1}(1 - h\omega\tau\omega T - k\omega^2T^2) \\ &- \delta_{P,n-1}(h\omega\tau\omega T + k\omega^2T^2) \\ &+ \omega^{-1}\dot{x}_{n-1}[\omega\tau + (\ell - 2\zeta h\omega\tau)\omega T \\ &- 2\zeta k\omega^2T^2]\}\end{aligned} \tag{2.16}$$

We now have the three difference equations (2.9a) (2.9b) and (2.16) for the three unknowns x_n, $\dot{x}_n$ and δ_{Pn}. Their solution will provide the result of our model of the hybrid calculation.

Solution by z-transform

The z-transform provides a simple method of solving the difference equations. The definition of the z-transform of the sequence x_n is

$$x^* = \sum_{n=0}^{\infty} x_n z^{-n} \tag{2.17}$$

and for our purposes its important property is

$$\sum_{n=0}^{\infty} x_{n+1} z^{-n} = z(x^* - x_0) \tag{2.18}$$

The inversion of a z-transform follows easily by observing from the definition (2.17) that

$$z^{k-1} x^* = \sum_{n=0}^{\infty} x_n z^{k-n-1}$$

If this is looked upon as a Laurent expansion in the complex variable z the residue is the coefficient of the term for which n = k, which is x_k. Thus the inversion of x^* to find x_n is accomplished by finding, for each n,

$$\text{Residue}\,(z^{n-1} x^*) = x_n \tag{2.19}$$

The stability of the solution is also indicated by (2.19). Stability requires that x_n not grow as n increases. The only factor in the residue which depends on n is z^n, which grows or decreases with n depending on whether the absolute value of z is greater or less than unity. This leads to the well-known stability criterion that every root of the denominator of x^* must have absolute value equal to or less than unity.

The transformation of (2.9) and (2.16) is accomplished by multiplying by z^{-n} and z^{-n+1} respectively, summing and using (2.17) and (2.18), remembering the initial conditions are zero. The result is

$$\begin{aligned}(z - 1)x^* - \dot{x}^*(e^{-\omega\zeta T}/\omega\zeta^1)\sin\omega\zeta^1 T \\ + (x^* - \delta_P^*)\,[1 - e^{-\omega\zeta T}(\cos\omega\zeta^1 T \\ + \zeta/\zeta^1 \sin\omega\zeta_1 T)] = 0\end{aligned} \tag{2.20a}$$

$$\begin{aligned}\dot{x}^*\,[z - e^{-\omega\zeta T}(\cos\omega\zeta^1 T - \zeta/\zeta^1 \sin\omega\zeta^1 T)] \\ + (x^* - \delta_P^*)\,\omega/\zeta^1\, e^{-\omega\zeta T}\sin\omega\zeta_1 T = 0\end{aligned} \tag{2.20b}$$

$$\begin{aligned}(z + K)x^* + \dot{x}^*K\omega^{-1}\,[\omega\tau + (\ell - 2\zeta h\omega\tau)\omega T \\ - 2\zeta k\omega^2T^2] - (x^* - \delta_P^*)\,[z + K\,(h\omega\tau\omega T \\ + k\omega^2T^2)] = z^2Kx_c/(z - 1)\end{aligned} \tag{2.20c}$$

These equations have been arranged so the variables are the z-transforms x^*, $\dot{x}^*$, and $x^* - \delta_P^*$, and their solution gives the z-transforms of the problem variables, which must then be inverted to yield formulas for the actual solution.

If the three equations are solved by determinants the denominator is given by the determinant of the coefficients,

$$\begin{aligned}\Delta = &- z\,[(z-1)^2 - 2z(e^{-\omega\zeta T}\cos\omega\zeta^1 T - 1)\\ &+(e^{-2\omega\zeta T}-1)] + K\{(z+1)(e^{-\omega\zeta T}\cos\omega\zeta^1 T - 1)\\ &-(e^{-2\omega\zeta T}-1) + (z-1)1\,\zeta^1 e^{-\omega\zeta T}\sin\omega\zeta^1 T\\ &[\zeta - \omega_T - (\ell - \zeta h\omega_T)\omega T + \zeta k\omega^2 T^2]\\ &-(z-1)[(z-1) - (e^{-\omega\zeta T}\cos\omega\zeta_1 T - 1)]\\ &(h\omega_T\omega T + k\omega^2T^2)\}\end{aligned} \tag{2.21}$$

This is a cubic in z, whose roots determine the solution through their residues, according to (2.19).

The solution for x^* is then

$$x^* = \frac{z^2\,Kx_c}{(z-1)\,\Delta}[(z+1)(e^{-\omega\zeta T}\cos\omega\zeta^1 T - 1) - (e^{-2\omega\zeta T}-1) + (z-1)\,\zeta/\zeta^1\,e^{-\omega\zeta T}\sin\omega\zeta^1 T] \tag{2.22}$$

An additional root at $z = 1$ is visible here, whose residue also makes a contribution.

Expansion of roots

The nature of the roots of Δ can be seen by letting T approach zero in (2.21). Then all terms approach zero except the first, so one root must approach zero, the other two approach unity. The exact roots are complicated to find since (2.21) is cubic, but we can be satisfied with expansions of the roots in powers of ωT.

Let us first look for a root of the form:

$$z = 1 + d\omega T + e\omega^2T^2 + f\omega^3T^3 + \cdots \tag{2.23}$$

If the coefficients of (2.21) are also expanded in powers of ωT, and (2.23) is inserted, setting the lowest two powers of ωT to zero yields

$$d^2 + (2\zeta + K_T)d + (1 + K) = 0 \tag{2.24}$$

$$e = \tfrac{1}{2}d^2 - \frac{Kd[d\omega_T(h - 3/2) + (\ell - 3/2)]}{2(d + \omega_T\,\zeta_T/\omega)} \tag{2.25}$$

These determine the first two coefficients in (2.23). The solution of (2.24) is

$$d_{1,2} = \omega_T(-\zeta_T \pm i\zeta^1{}_T)/\omega = \lambda_{T1,2}/\omega \tag{2.26}$$

where ω_T, ζ_T, λ_T are defined in (2.3) and (2.4). Thus the first coefficient is identical with the exponent of the exact solution.

To see the significance of this, remember that the important term in the residue is z^n which can be written $\exp(n\,\ell n\,z)$. But z in the form (2.23) can be used to expand $\ell n\,z$ to yield

$$\begin{aligned}z^n = \exp\{&nd\omega T + n(e - d^2/2)\omega^2T^2\\ &+ n[f - d^3/6 + d(e - d^2/2)]\,\omega^3T^3 + \cdots\}\end{aligned} \tag{2.27}$$

Thus the first term is part of the exact solution at $t = nT$, and subsequent terms are error terms.

With two roots z_1, z_2 given as a complex conjugate pair by (2.23)–(2.26), the third root is simple to find by dividing Δ by $(z - z_1)\,(z - z_2)$. The expanded result is, using (2.24) and (2.25),

$$\begin{aligned}z_3 = &(1 - h)\,K\omega_T\,\omega T\\ &+ K[\ell - k - 1/2 + \zeta\omega_T(1 - 2h)\\ &+ K(1 - h)\,\omega^2\tau^2]\omega^2T^2 + \cdots\end{aligned} \tag{2.28}$$

The solution is usually stable to the roots z_1, z_2 because the real part of d is negative, so the dominant term of z^n is a damping. However, it may be unstable to z_3, and will be for large enough ωT.

Before choosing values for the compensation parameters, we will look at the actual solution generated by these roots.

Solution in the physical (time) domain

The solution is the sum of the residues of $(z^{n-1}\,x^*)$ at the poles $z = 1, z_1, z_2, z_3$, with x^* given by (2.22). The residue at $z = 1$ is easily found by putting $z = 1$ into $(z - 1)\,x^*$, which yields

$$\text{Residue }(z = 1) = Kx_c/(1 + K) \tag{2.29}$$

which is just the constant part of the exact solution (2.4).

Since z_1 and z_2 are complex conjugates, so are their residues, and their sum is twice the real part of either. If the expansion (2.23) is put into (2.22) and (2.21), the result for z_1 to order ωT is found to be

$$\text{Residue }(z_1) = \frac{z_1{}^n\,Kx_c}{2(1 + K)}\left[-\left(1 + \frac{i\beta_r\omega T}{\omega_T\zeta_T/\omega}\right) + \frac{i\zeta_T}{\zeta_{1T}}\left(1 - \frac{\beta_i\omega T}{\omega_T\zeta_T/\omega}\right)\right] \tag{2.30}$$

$$e_1 - d_1{}^2/_2 \equiv \beta_r + i\beta_i$$

Finally, the residue at z_3 is found similarly using (2.28):

$$\text{Residue } (z_3) = 2^{-1}(\omega T)^{n+3}\,[K\omega_T(1 - h)]^{n+1} \qquad (2.31)$$

Choice of compensation constants

Comparison of (2.27) and (2.29) with the exact solution (2.4) shows that the first deviation of both the z^n factor, and the rest of the expression, depend on $e - d^2/2$. If this term is zero, the deviation will then be $0(\omega^2T^2)$ in both places. And (2.31) shows that the contribution of the extraneous solution is of high order in ωT and should decrease rapidly as long as $|z_3| < 1$.

These observations lead to the conclusion that we should make $e - d^2/2$ vanish, which means, according to (2.25),

$$h = \ell = 3/2 \qquad (2.32)$$

The coefficient k is not determined to this order. However, if $e - d^2/2 = 0$ the next term in (2.27) is found from the expansion of (2.21) to be

$$f - d^3/6 = \frac{-Kd\{d[k - 13(1 + d\omega_T)/12] - 2K\omega_T(1 + d\omega_T)/3\}}{2(d + \omega_T\zeta_T/\zeta)}$$

This cannot vanish for any choice of real k. One can make either its real part or its imaginary part vanish, although k will then depend on the parameters of the problem. One obvious choice which reduces the size of $f - d^3/6$ is

$$k = 13/12 \qquad (2.33)$$

and this is the one used in the implementation. Further study would be needed to determine if another, more complicated, choice were better.

Notice that the values given in (2.32) are exactly those shown in (2.15), which are obtained by projecting to $(n + 1/2)T$, while the k of (2.33) is only 1/24 smaller than the corresponding value of k in (2.15). One can therefore look upon the analysis as providing a derivation of the length of the projection interval, in contrast with the usual graphical or intuitive arguments.

Results for three-step compensation

An entirely analogous solution can be obtained using the three-step projection of (2.13). The necessary starting values δ_{-1} and δ_{-2} are taken the same as δ_0. The determinant of the coefficients is now fifth degree, with five roots. Two are of the form (2.23) with d the same, (2.24). The next coefficients are

$$e - \tfrac{1}{2}\,d^2 = \frac{Kd(1 + d\omega_T)(a_1 + 2a_2 + 3/2)}{2(d + \omega_T\zeta_T/\zeta)} \qquad (2.34)$$

and, if $e - d^2/2 = 0$,

$$f - d^3/6 = \frac{-K(1 + d\omega_T)[Kd\omega_T/12 + d^2(a_2 - 22/12)]}{2(d + \omega_T\zeta_T/\zeta)} \qquad (2.35)$$

The other three roots are power series in $(\omega T)^{1/3}$, given in terms of

$$r = (-1 + i3^{1/2})/2, \quad \bar{r} = (-1 - i3^{1/2})/2$$

by

$$\begin{aligned} z_{3,4,5} = {} & (K\omega_T a_2\omega T)^{1/3}(1, , \bar{r}) \qquad (2.36) \\ & + (K\omega_T a_2\omega T)^{2/3}(a_1 + a_2)(1, \bar{r}, r)/3a_2 \\ & + K\omega_T a_2\omega T/3a_2 + \cdots \end{aligned}$$

The residues at $z = 1$ and $z = z_1$ are the same as for Taylor series compensation, (2.29) and (2.30). The first terms of the residues of the other three roots are

$$\begin{aligned} \text{Residue } (z_{3,4,5}) = {} & (Kx_c/6)(\omega T)^{(n+7)/3} \\ & (K\omega_T a_2)^{(n+1)/3}\,(1, r, \bar{r})^{n+1} \qquad (2.37) \end{aligned}$$

To make the $0(\omega T)$ errors vanish we make $e - d^2/2 = 0$ by taking

$$a_1 + 2a_2 = -3/2$$

which agrees with (2.14). To determine a_1, a_2 separately one can go to (2.35) and choose $a_2 = 22/12$, which is 1/24 less than the value in (2.14). So again we come very close to the usual projection distance by an analytical derivation.

The error caused by the extraneous roots should not be as small for this type of compensation, since it depends on $(\omega T)^{n/3}$, and decreases rather slowly, as n increases.

The solution is also less stable, because of the one-third power dependence of the roots on ωT. In fact, the absolute values through the first two terms are

$$|z_{3,4,5}| = (K\omega_T a_2 \omega T)^{1/3}$$
$$|1 + (1, -\tfrac{1}{2}, -\tfrac{1}{2})(K\omega_T a_2 \omega T)^{1/3}\,(a_1 + a_2)/3a_2| \tag{2.38}$$

and since $a_1 + a_2$ is negative, the conjugate pair z_4, z_5 is the least stable. This is the pair introduced by going back two steps in time, which shows the destabilizing influence of that procedure.

Stability considerations

As mentioned already, it is the extraneous roots which control the stability of the hybrid calculation. For the Taylor series compensation, this root is given by (2.28), and is of the order $K\omega_T\omega T$, the same as for the uncompensated case, which can be obtained from (2.28) by putting $k = h = \ell = 0$. In fact, the compensated root is somewhat smaller (thus more stable) since the coefficient of the first term is $-1/2$ instead of 1. Notice that one could improve the stability, at some cost in accuracy, by choosing k so that the coefficient of the second term in z_3 vanishes, although k would then depend on the parameters of the problem instead of being constant.

In contrast, the extraneous roots for three-step compensation are given in (2.38) and are of order $(a_2\ K\omega_T\omega T)^{1/3}$, considerably larger than the uncompensated or Taylor series cases. Therefore, the three-step method yields a less stable solution. If $a_2 = 0$, we then have a two-step scheme, and there are only two extraneous roots, of order $(K\omega_T\omega T)^{1/2}$, more stable then the three-step scheme but still less stable than the uncompensated or Taylor series cases.

If the scheme of Howe and Fogarty, discussed in the Introduction, were used, there would also be two extraneous roots of order $(K\omega_T\omega T)^{1/2}$, so the stability would be about the same as for a two-step scheme. In fact, the two-step and Howe-Fogarty schemes are closely related, both going back one step in time.

Computer implementation

The Taylor series (or derivative) method of compensation was tested, and compared with the three-step method, by solving the problem posed by (2.1), (2.2) on the hybrid computer of the NASA Electronics Research Center. This is a Beckman 2200/SDS 9300 machine with interface built by Beckman.

As described above, integration the of x and x, and the combination of x and x on the left side of (2.1) were performed in the analog computer. The value of δ was found in the digital computer, by sampling x and x from the analog at intervals of T and extrapolating. Then δ_P was calculated and sent back to the analog to be used to find x. The A to D sampling was accomplished first, followed immediately by the D to A updating. In order to compare the resulting hybrid solution with a continuous solution, the complete equation was also solved in the analog simultaneously as an oscillator with frequency ω_T and damping ζ_T as defined by (2.3). The details of the analog circuit, the digital programs, the control circuit, the scaling, etc., are given in Ref. 7, pages 130-141 and Appendix E.

The output of this calculation was a set of strip-charts and digital printouts giving the hybrid and pure analog values of x, $\dot{x}$, x, δ, and the difference between the hybrid and analog values, which may be taken as a measure of the error of the hybrid solution.

Runs were made for the parameters

$$\omega = 0.412,\ \zeta = -0.2425,\ \zeta_T = 0.7$$

using the conventional compensation constants

$$a_1 = -21/4,\ a_2 = 15/8,\ a_0 = 1 - a_1 - a_2 = 35/8$$

for the three-step method, and the set

$$\ell = h = 3/2,\ k = 13/12$$

which we have derived for the Taylor series method. The values of ω_T were varied between 0.5 and 15.0. For each such value, the control parameters K and τ can be calculated from (2.3). Runs were made at several sample intervals T in order to study the stability of the hybrid calculation. For large enough T is was always possible to make it unstable.

The relative merits of the Taylor series and three-step compensation schemes, compared to pure analog and uncompensated hybrid results, are strikingly illustrated by excerpts from the strip charts drawn by the analog computer. The case chosen for illustration is $\omega_T = 15$, for which (2.3 gives $K = 1234$, $\tau = 0.0942$.

Figure 1 shows the strip chart record for x(t) for four cases. At the top is the continuous solution produced by a pure analog calculation. Below follow the records for the uncompensated, Taylor series compensated, and three-step compensated hybrid solutions all for a sample interval T = 25 milliseconds, which is 17 samples per cycle based on total frequency. In order to bring out the errors more clearly, Figure 2 shows the difference signal $x_H - x_A$ on a larger scale, where the subscripts H and A stand for hybrid and analog, re-

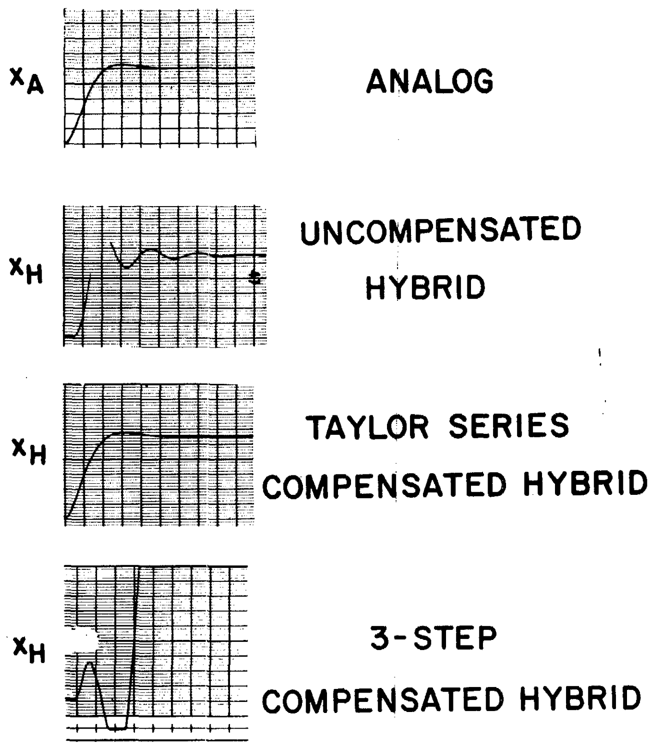

Figure 1—Strip chart records of x(t) for $\omega = 0.412$, $\zeta = -0.2425$, $\zeta_T = 0.7$, $\omega_T = 15$. The sample interval T = 25 ms.

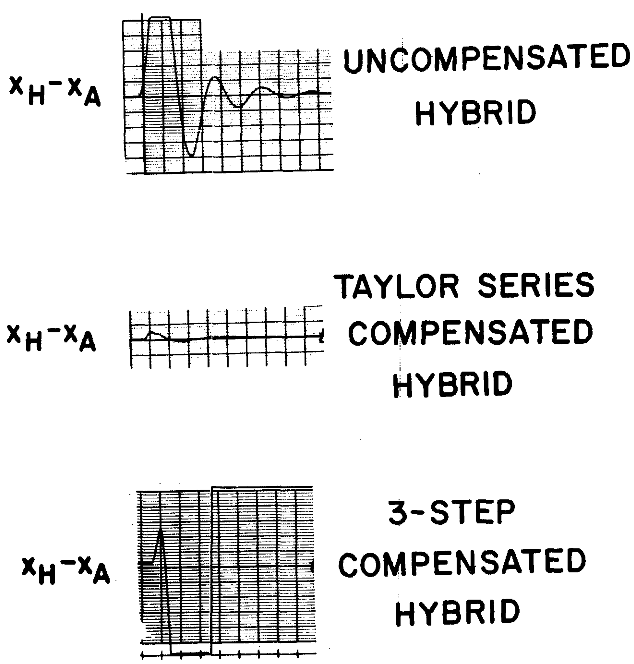

Figure 2—Strip chart records of x_H (HYBRID) $-x_A$ (ANALOG) for $\omega = -0.412$ $\zeta = 0.2425$, $\zeta_T = 0.7$, $\omega_T = 15$. The sample interval T = 25 ms.

spectively. The great improvement in accuracy achieved by going from no compensation to Taylor series to compensation is apparent. On the other hand, the solution with three-step compensation is unstable and saturates the amplifiers.

The stability properties of these three cases are predicted by the formulas we have developed. For no compensation ($\ell = h = k = 0$), (2.28) gives $|z_3| = 0.736$, while for Taylor series compensation the same formula shows $|z_3| = 0.411$. On the other hand, for three-step compensation, (2.38) gives $|z_3| = 0.578$, $|z_4, z_5| = 1.375$. Therefore, the part of the solution corresponding to the root z_3 is stable, but the part corresponding to the roots z_4, z_5 are unstable, leading to an unstable solution, as shown in Figures 1 and 2. To stabilize the three-step case, the sample interval T would have to be reduced to 10 ms, or about 42 samples per cycle, for which (2.38) shows $|z_4| = 0.928$. A case run at this value of T indeed showed three-step compensation to yield a stable solution.

To destabilize the uncompensated and Taylor series cases, a run was made at T = 50 ms (8.4 samples per cycle), for which (2.28) gives $|z_3| = 1.89$ and 1.12, respectively. The results of the run are shown in Figure 3, where the rapid increase of x until the amplifiers saturate is seen for both cases.

Similar results hold for other values of ω_T. In all cases, stability or instability exhibited by the numerical calculations could be predicted in advance by use of (2.28) or (2.38). Furthermore, the digital printouts showed that with the same set of parameters and sampling interval, the Taylor series method gave more accurate results, that is, results closer to the analog (continuous) solution. The improvement in accuracy

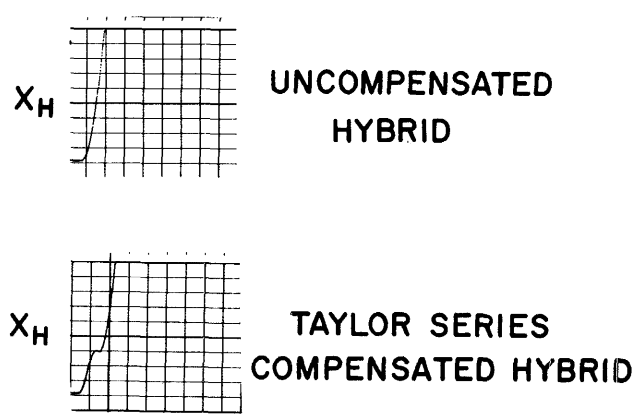

Figure 3—Strip chart records of x(t) for $\omega = 0.412$ $\zeta = -0.2425$, $\zeta_T = 0.7$, $\omega_T = 15$. The sample interval T = 50 ms.

could be quite marked for sample intervals near the stability limit of the three-step method. This is in accord with the deductions from the extraneous solutions (2.31) and (2.37).

CONCLUSIONS

The Taylor series (or derivative) method of compensation appears to have a number of advantages over the three-step method of compensation for the time delay and D to A hold errors of hybrid computing. For a given case, it can be made stable for larger sampling intervals than the three-step method, and is more accurate at the same sampling interval. The Taylor series method can also be made stable for larger sampling intervals than the uncompensated case for almost all values of the parameters, while the three-step method may well be unstable when the uncompensated calculation is stable. In other words, compensating by Taylor series can improve the stability, while compensating by the three-step method destabilizes.

These stability advantages of the Taylor series method depend to a large extent on the particular form of implementation used. The crucial point is not to use information which goes back in time, since each such backward time step introduces an extraneous solution which is de-stabilizing. The implementation suggested here, where the extrapolations are accomplished in the digital computer, avoids extra backward time steps while still permitting the A to D sampling to be done before the D to A transfer. If the extrapolations are done in the analog computer, as in the Howe-Fogarty[5,6] implementation, the A to D before D to A sequence of operations introduces one backward time step and adversely effects the stability. If the sequence is performed in the order D to A followed by A to D, the analog extrapolation of Howe and Fogarty would give exactly the results of the present analysis.

The z-transform method of analysis for linear equations can be carried through with arbitrary coefficients in the extrapolation formulas. Then they can be chosen to yield the desired improvement in accuracy and/or stability. The coefficients of the first power of the sample interval T clearly should be chosen to extrapolate by 1.5T, but the coefficient of T^2 in the extrapolation formula for x has some flexibility in the choice, depending on whether accuracy or stability is the paramount consideration.

When the derivatives are available, there is no more difficulty implement the Taylor series method than the three-step method, and there are no starting problems with the former, as there are with the latter.

On the basis of the analysis and numerical results of this study, the Taylor series method of compensation seems preferable in all ways to the three-step method, and can be recommended whenever the derivatives are available. Whether this conclusion also will hold for non-linear equations and for higher order systems, depends on the results of applying the Taylor series method to those cases. Some preliminary study of a linear fourth order system by the present method of analysis indicates that the Taylor series method may be applicable, but with d fferent values of the compensation coefficients.

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