UNIVERSITY OF ILLINOIS

DIGITAL COMPUTER

LIBRARY ROUTINE E5 - 195

TITLE	Quadrature by Gauss' Method (DOI or SADOI)
TYPE	Closed
NUMBER OF WORDS	18 + 2n
TEMPORARY STORAGE	1 word at 0 (may also be used by auxiliary subroutine)
DURATION	n(1.8 + T) milliseconds, where T is the duration of the auxiliary subroutine.
ACCURACY	See Note.
ENTRY	When this routine is located at q entry is made by

p	-- sF
	50 pF
p + 1	26 qF

where s is the location of the auxiliary subroutine which computes values of $f(u_i)$.

DESCRIPTION

This routine uses the method of Gauss for approximating the integral:

$$\int_0^1 f(u)du \approx \sum_{i=1}^{n} a_i f(u_i) \qquad (1)$$

The values $f(u_i)$ are obtained from an auxiliary subroutine which must be prepared by the user. The auxiliary subroutine is entered in the standard manner with u_i in A, and it should place $f(u_i)$ in A. The values of the u_i are in the range $0 < u_i < 1$. Location 0 is used by this routine to store the new value of $f(u_i)$ each time control is

returned to this routine by the auxiliary routine.

At the end of the library copy of this routine are sections of tape containing constants and the values of u_i and a_i for each value of n from 2 through 16. The choice of n is made by copying the appropriate section of tape after copying the part containing the basic routine.

ACCURACY

Quadrature is carried out numerically by approximating the integral with a finite summation

$$\int_a^b g(x)dx \approx L \sum_{i=1}^{n} a_i f(u_i) \qquad (2)$$

where

$$L = b - a$$

$$u = (x - a)/L$$

$$f(u) = g(x) = g(Lu + a)$$

Gauss' Method of quadrature consists essentially of making an optimum choice of the values of both the u_i and the a_i. When the accuracy with which values of $g(x)$ can be obtained is independent of x, the method of Gauss gives the highest possible order of accuracy for a given number n of ordinates. When n ordinates are used, the order of accuracy achieved is $2n - 1$; i.e. the truncation error is zero for polynomials of degree $2n - 1$ or less. For an arbitrary function having a continous $2n^{th}$ derivative the

truncation error is, for the integral shown in equation (1),

$$R_n(F) = \frac{f^{(2n)}(\xi)}{(2n+1)!} \left[\frac{(n!)^2}{(2n)!}\right]^2$$

where ξ is some point in [0,1], and for the integral shown in equation (2)

$$R_n(g) = (b-a)^{2n+1} \frac{g^{(2n)}(\xi)}{(2n+1)!} \left[\frac{(n!)^2}{(2n)!}\right]^2$$

where ξ is some point in [a,b].

In addition to the truncation error there is a round-off error which may be significant. It is given by (see equation (1))

$$\epsilon = \sum_{i=1}^{n} [a_i \Delta f(u_i) \pm 2^{-40} f(u_i) \pm 2^{-40} a_i f'(u_i)] \pm 2^{-40}$$

$$\leq \pm 2^{-40} \max_{0 \leq u \leq 1} (n|f(u_i)| + |f'(u_i)|) + \Delta \pm 2^{-40}$$

where $\Delta = \max \Delta f(u)$ is the accuracy of the auxiliary subroutine. The terms $2^{-40} f(u_i)$ and $\pm 2^{-40} a_i f'(u_i)$ represent errors resulting from a maximum error of 2^{-40} in the value of each of the a_i and u_i, respectively. The last term, 2^{-40}, represents the final round-off of the double precision sum of products. The total error is bounded by the sum of the truncation error and the round-off error.

EXAMPLE

Using n = 5 ordinates in approximating the

integral

$$\int_0^1 \frac{1}{2} \sin \pi x dx = \frac{1}{\pi}$$

this routine gave a value of
0.31830990373, which is off 1 unit
in the 8^{th} decimal place.
Using the same number of ordinates integration
by Simpson's rule (code E-3) gave a value of
0.31903559373, which is off 1 unit in the
third decimal place.

Rt: 7/20/59

DATE 9/16/55 Rt: 6/3/58
CODED BY Carl Farrington
APPROVED BY J. P. Nash

LOCATION	ORDER		NOTES PAGE 1
	00 K(E5)		
0	L5 16L		Set addresses
	46 5L		
1	42 7L		
	K5 F		
2	46 6L		Plant S
	42 13L		Plant link
3	41 14L		Set $\Sigma = 2^{-40}$
	2S 4L		
4	S5 F	from 12	
	40 15L	from 3	
5	L5 F	by 0, 10	u_i
	50 5L		Enter aux. routine
6	26 F	by 2	
	40 F		$f(u_i)$
7	L5 15L		i
	50 F	by 1, 11	$\sum_{i=1} a_i f(u_i)$
8	74 F		
	L4 14L		
9	40 14L		
	L5 8L		Step addresses
10	L4 5L		
	46 5L		
11	F5 7L		
	42 7L		
12	L0 17L		Test i > N
	36 4L		
13	L5 14L		$\int f(u)du$
	22 F	by 2	link
14	00 F		
	00 F		
15	00 F		
	00 F		

LOCATION	ORDER	NOTES	PAGE 2
16	00 18L		
	00 (n+18)L		
17	75 15L		
	50 (2n + 18)L		
18	u_1		
19	u_2		
	.		
	.		
	.		
	.		
	.		
n + 17	u_n		
n + 18	a_1		
n + 19	a_2		
	.		
	.		
	.		
	.		
	.		
	.		
2n + 17			
	a_n		