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P. Erdös
R. L. Graham
E. Szemerédi

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P. Erdös*, R. L. Graham**, E. Szemerédi*
Stanford University
Stanford, California

INTRODUCTION

The following problem was raised by H.- J. Stoss [3] in connection with certain questions related to the complexity of Boolean functions. An acyclic directed graph G is said to have property $\mathcal{P}(m,n)$ if for any set X of m vertices of G, there is a directed path of length n in G which does not intersect X. Let $f(m,n)$ denote the minimum number of edges a graph with property $\mathcal{P}(m,n)$ can have. The problem is to estimate $f(m,n)$.

For the remainder of the paper, we shall restrict ourselves to the case $m = n$. We shall prove

(1) $$c_1 n \log n / \log \log n < f(n,n) < c_2 n \log n$$

(where $c_1, c_2, \ldots,$ will hereafter denote suitable positive constants). In fact, the graph we construct in order to

* P. Erdös and E. Szemerédi, Mathematical Institute of the Hungarian Academy of Sciences, Réaltanoda ut. 13-15 Budapest V, Hungary.

** R. L. Graham, Bell Laboratories, Department of Mathematics, 500 Mountain Avenue, Murray Hill, New Jersey.

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establish the upper bound on $f(n,n)$ in (1) will have just c_3n vertices. In this case the upper bound in (1)is essentially best possible since it will also be shown that for c_4 sufficiently large, every graph on c_4n vertices having property $\mathcal{P}(n,n)$ must have at least $c_5n \log n$ edges.

A PRELIMINARY LEMMA

In order to establish the upper bound in (1) we first need the following result.

Lemma. For all $\delta > 0$ there exists $c = c(\delta)$ such that for all t sufficiently large, there exists a bipartite graph $B = B(\delta;t)$ with vertex sets A and A' so that:

(i) $|A| = |A'| = t$;

(ii) B has at most $c(\delta)t$ edges;

(iii) If $X \subseteq A$, $X' \subseteq A'$ with $|X| \geq \delta t$, $|X'| > \delta t$ then $(X,X') = \{\{x,x'\}: x \in X, x' \in X'\}$ contains an edge of B.

Proof: We use a simple probabilistic argument to show the existence of B. Form a bipartite graph $\bar{B}$ on the vertex sets A and A' with $|A| = |A'| = t$ by selecting for each $a \in A$ a random subset $\bar{B}(a) \subseteq A'$ of cardinality $d = d(\delta)$ (to be specified later). Call $\bar{B}$ "bad" if there exists $X \subseteq A$, $X' \subseteq A'$, with $|X| \geq \delta t$, $|X'| > \delta t$, so that (X,X') contains no edge of $\bar{B}$. For fixed X and X', the probability that $\bar{B}$ is bad because of these two subsets is at most

$$\binom{(1-\delta)t}{d}^{\delta t} \Big/ \binom{t}{d}^{\delta t} < \left(\frac{(1-\delta)t}{t-d}\right)^{d\delta t}.$$

Hence, the total probability that $\bar{B}$ is bad is at most

$$\binom{t}{\delta t}^2 \left(\frac{(1-\delta)t}{t-d}\right)^{d\delta t} < 2^{2t}\left(\frac{1-\delta}{1-d/t}\right)^{d\delta t}$$

A simple computation shows that if d is chosen suitably large, for example, so that

$$(1-\delta^2)^{d\delta} < 1/4,$$

then for t sufficiently large this probability is less than 1, and so, a graph $B = B(\delta;t)$ must exist which satisfies the requirements of the lemma. ∎

CONSTRUCTION OF G

The next step in the proof of (1) is the construction of the directed graph G. For large n, $G = G(n)$ will have as its vertex set the set $V = \{0,1,\ldots,2^n-1\}$. If v and m are positive integers, then $D_v(m)$ will denote the set $\{v, v+1, \ldots, v+m-1\} \cap V$. Similarly, $D_v^*(m)$ will denote the set $\{v, v-1, \ldots, v-m+1\} \cap V$. In general, $\varepsilon_1, \varepsilon_2, \ldots,$ will denote suitably chosen fixed positive constants to be specified later. The edge set E of G is formed as follows:

(i) For $v \in V$, the pairs (v,x), $x \in D_{v+1}(4n)$, are in E;

(ii) For each t with $n/2 \le 2^t < 2^n$ and each i as specified below, a copy of $B(\varepsilon_i;2^t)$ is formed between the vertex sets $A = D_{m\cdot 2^t}(2^t)$ and $A' = D_{(m+i)\cdot 2^t}(2^t)$, $0 \le m < 2^{n-t}$, where $i = 1,2,\ldots,10$ (or if i cannot assume the value 10 because $(m+10)2^t > 2^n$, then it ranges from 1 to $2^{n-t}-m$). All edges are directed from x to y with $x < y$.

An elementary calculation shows that

$$|E| < c_6 n 2^n.$$

THE UPPER BOUND

Theorem 1. For a suitable $\varepsilon > 0$, $G(n)$ has property $\mathcal{P}(\varepsilon \cdot 2^n, \varepsilon \cdot 2^n)$ for all sufficiently large n.

Proof: The theorem will be proved by a sequence of claims. First we show that $G(n)$ shares with the graphs $B(\varepsilon;t)$ the following property.

Claim 1. If $m \geq 2n$ and $X \subset D_x(m)$, $X' \subset D_{x+m}(m)$, satisfy $|X| \geq \varepsilon_2 m$, $|X'| \geq \varepsilon_2 m$, then $[X,X'] = \{(x,x') : x \in X, x' \in X'\}$ contains an edge of $G(n)$.

Proof of Claim: Let $2^t \leq m/2 < 2^{t+1}$. Thus, $m/4 < 2^t$ so at most five of the intervals $D_{r \cdot 2^t}(2^t)$ intersect $D_x(m)$ and at most five of them intersect $D_{x+m}(m)$. Since $|X| \geq \varepsilon_2 m$ then some $D_{r \cdot 2^t}(2^t)$ and $D_{r' \cdot 2^t}(2^t)$ have

$$(3) \quad |D_{r \cdot 2^t}(2^t) \cap X| \geq \varepsilon_2 m/5, \quad |D_{r' \cdot 2^t}(2^t) \cap X'| \geq \varepsilon_2 m/5.$$

But we must have $|r'-r| \leq 10$ so that by the construction of $G(n)$ there is a copy of $B(\varepsilon_1;2^t)$ between $D_{r \cdot 2^t}(2^t)$ and $D_{r' \cdot 2^t}(2^t)$. Thus, if $\varepsilon_2/5 > \varepsilon_1$ and $m > 2^t$ then the property of $B(\varepsilon_1;2^t)$ guaranteed by the Lemma implies that $[X,X']$ contains an edge of $G(n)$ provided that t is sufficiently large (which is guaranteed by choosing n large enough). This proves the claim. ■

Next, let us choose an arbitrary fixed set X of vertices with $|X| \leq \varepsilon \cdot 2^n$. The vertices in X will be referred to as the marked vertices of G; the remaining vertices of G will be called the unmarked vertices of G.

Let us call an unmarked vertex $y \in V$ bad if for some $m \geq 1$ either at least $\varepsilon_3 m$ vertices in $D_y(m)$ are marked or at least $\varepsilon_3 m$ vertices in $D_y^*(m)$ are marked. Otherwise, an unmarked vertex of G is called good.

Claim 2. There are at *most* $\varepsilon_4 2^n$ bad vertices.

Proof of Claim: Let y_1 denote the least unmarked vertex of G (if it exists) for which for some $m_1 \geq 1$, at least $\varepsilon_3 m_1$ vertices in $D_{y_1}(m_1)$ are marked. In general, if $y_1, \ldots, y_k$ and $m_1, \ldots, m_k$ have been defined, let y_{k+1} be the least unmarked vertex of G following $y_k + m_k - 1$ (if it exists) for which for some $m_{k+1} \geq 1$ at least $\varepsilon_3 m_{k+1}$ vertices in $D_{y_{k+1}}(m_{k+1})$ are marked. We continue this process until it no longer can be applied, so that, say, $y_1, \ldots, y_s$ and $m_1, \ldots, m_s$ have been defined. Similarly, let y_1^* denote the greatest unmarked vertex (if it exists) for which for some $m_1^* \geq 1$, at least $\varepsilon_3 m_1^*$ vertices in $D_{y_1^*}^*(m_1^*)$ are marked, etc. In this way, we define $y_1^*, \ldots, y_{s^*}^*$ and $m_1^*, \ldots, m_{s^*}^*$.

It follows from the preceding construction and the definition of a bad vertex that all bad vertices are contained in the set

$$Y = \bigcup_{k=1}^{s} D_{y_k}(m_k) \cup \bigcup_{k=1}^{s^*} D^*_{y^*_k}(m^*_k)$$

Thus, there are at most

$$M = \sum_{k=1}^{s} m_k + \sum_{k=1}^{s^*} m^*_k$$

bad vertices. However, by our construction there are at least $(\varepsilon_3/2)M$ marked vertices in Y. Since by hypothesis there are at most $\varepsilon \cdot 2^n$ marked vertices in V then we have

$$(\varepsilon_3/2)M \leq \varepsilon \cdot 2^n,$$

$$M \leq (2\varepsilon/\varepsilon_3)2^n < \varepsilon_4 2^n,$$

which proves the claim. ■

For an unmarked vertex x, let P (m) denote the set of all unmarked vertices in $D_x(m)$ which can be reached from x by directed paths which contain only unmarked vertices.

Claim 3. If x is a good vertex and $|D_x(m)| = m$ then

(4) $$|P_x(m)| > \varepsilon_5 m$$

Proof of Claim: If $m \leq 4n$ then since x is good, at least $(1-\varepsilon_3)m$ vertices in $D_x(m)$ are unmarked and x has edges directly to all of them. Suppose $m > 4n$. Let m' denote $[m/2]$. Since $|D_x(m')| = m'$ then by induction $|P_x(m')| > \varepsilon_5 m'$. Since x is good then at most $\varepsilon_3 m$ vertices in $D_x(m)$ are marked. Hence, at most $\varepsilon_3 m$ vertices in $D_{x+m'}(m') \subseteq D_x(m)$ are marked. Since $m' \geq 2n$ and $\varepsilon_5 \geq \varepsilon_2$ then there are edges from $P_x(m')$ to at least $(1-\varepsilon_2)m'$ vertices in $D_{x+m'}(m')$. But at most $\varepsilon_3 m < 3\varepsilon_3 m'$ vertices in $D_{x+m'}(m')$ are marked. Hence, $P_x(m')$ must have edges to at least $(1-\varepsilon_2-3\varepsilon_3)m'$ unmarked vertices in $D_{x+m'}(m')$. Since $1-\varepsilon_2-3\varepsilon_3 > 3\varepsilon_5$ then

$$|P_x(m)| > 3\varepsilon_5 m' > \varepsilon_5 m.$$

The claim now follows by induction, □

In exactly the same way it follows that if $P_x^*(m)$ denotes the set of all unmarked vertices in $D_x(m)$ which are connected to the unmarked vertex x by a directed path containing only unmarked vertices, and x is a good vertex and $|D_x^*(m)| = m$, then-

(4′) $$|P_x^*(m)| > \varepsilon_5 m.$$

Claim 4. Let x and x′ be good vertices with $x < x'$. Then $x' \in P_x(2^n)$.

Proof : If $x'-x \leq 4n$ then the claim is immediate since by construction there is an edge from x to x′. Assume $x'-x > 4n$. Let $y = \lceil (x+x')/2 \rceil$ and let $m = y - x + 1$.

Consider the intervals $D_x(m)$ and $D^*_{x'}(m)$. Either they are adjacent or they have the single element y in common. Since x and x′ are good then by (4) and (4′)

$$(5) \qquad |P_x(m)| > \varepsilon_5 m, \ |P^*_{x'}(m)| > \varepsilon_5 m.$$

Since $\varepsilon_5 > \varepsilon_2$ then by Claim 1, there is an edge in G from a vertex of $P_x(m)$ to a vertex of $P^*_{x'}(m)$. Thus, there is a directed path from x to x′ containing no marked vertices and the claim is proved. ■

The proof of the theorem is now immediate. By Claim 2 there are at least $(1-\varepsilon_4-\varepsilon)2^n$ good vertices in G. By Claim 4 we can form a directed path which contains only unmarked vertices and which contains all the good vertices (since x′ can always be chosen to be the next good vertex following x). Since $1-\varepsilon_4-\varepsilon > \varepsilon$ then the theorem follows (where it is easily seen how the appropriate values of ε_k and c_k can be chosen). ■

THE LOWER BOUND

The following result will establish the lower bound in (1).

Theorem 2. Let H be an acyclic directed graph with at most $c_7 n \log n/\log\log n$ edges where n is a large fixed integer. Then there is a set of at most n vertices of H which hits every directed path of length n.

Proof: Let us denote the vertex set of H by $V = \{1,2,\ldots,v\}$. We may assume that H has at least $c_8\, n \log n/\log\log n$ edges. We may also assume that all edges are of the form (i,j) with

$i < j$. For an edge $e = (i,j)$ of H, let length (e) be defined to be $j-i$. Partition the edges of H into classes $C_0, C_1, \ldots, C_r$ where

$$C_k = \{e : 2^{4k} \log\log n < \text{length}(e) < 2^{4(k+1)} \log\log n\}$$

and $r = [\log v/4 \log\log n]$.

Since H has at least $c_8\, n \log n/\log\log n$ edges then it follows that $v \geq c_9\, n^{1/2}$ and $r \geq c_{10} \log n/\log\log n$. Hence some class C_a with $0 \leq a < r$ has at most $c_{11}\, n$ elements, Let us delete all vertices in H incident to any of the edges in C_a. Furthermore, we also delete those vertices $x \in V$ which satisfy

$$0 \leq x - m \cdot 2^{4a} \log\log n\,(1+2^{2} \log\log n) < 2^{4a} \log\log n$$

for some integer $m \geq 0$. This latter step removes at most

$$\left(\frac{2}{2^{2 \log\log n} - 1}\right) v = o(n)$$

vertices, since $v \leq 2\, c_7\, n \log n/\log\log n$. Hence we have deleted at most $c_{12}\, n$ vertices altogether. However, any directed path remaining has at most

$$\left(\frac{2^{(4a+2)\log\log n} - 2^{4a} \log\log n}{2^{4(a+1)} \log\log n}\right) v = o(n)$$

edges, since we cannot go more than $2^{(4a+2)} \log \log n - 2^{4a} \log \log n$ steps without using an edge whose length exceeds $2^{4a} \log \log n$; and the length of such an edge actually exceeds $2^{4(a+1)} \log \log n$. This proves the theorem. ■

By using a different partition of the edges of H, namely, into the classes $C_0', \ldots, C_r'$, where

$$C_k' = \{e : 2^{c_{13}k} \leq \text{length}(e) < 2^{c_{13}(k+1)}\}$$

for a suitable constant c_{13}, we can establish the following result.

<u>Theorem 3</u>. If c_{14} is sufficiently large then any graph G on $c_{14}n$ vertices having property $\rho(n,n)$ must have at least $c_{15}n \log n$ edges.

The graphs G(n) used in Theorem 1 show that the result in Theorem 3 is best possible to within constant factors.

<u>SOME RELATED QUESTIONS</u>

We now consider several problems for ordinary (undirected) graphs. Let $F_e(n,n)$ (resp., $F_v(n,n)$) denote the smallest integer for which there is a graph with $F_e(n,n)$ (resp., $F_v(n,n)$) edges so that with the deletion of any n of its vertices there still remains a connected component of n edges (resp., vertices). We shall prove by probabilistic methods that

(6) $$F_e(n, n) < c_{16}n, \; F_v(n, n) < c_{17}n.$$

The method we use is the same as that in the work of Erdös and Rényi [1], [2]. It turns out that almost all graphs have the desired property.

<u>Theorem 4</u>. For every $\varepsilon > 0$ there is a $c = c(\varepsilon)$ so that all but $o\left(\binom{\binom{(2+\varepsilon)n}{2}}{cn}\right)$ graphs G with $(2+\varepsilon)n$ vertices and cn edges have the property that after the omission of any n of its vertices, a connected component of at least n vertices remains.

<u>Proof</u>: It suffices to show that if n vertices are omitted and the remaining $n(1+\varepsilon)$ vertices are split into two classes S_1 and S_2 with $|S_1| > \varepsilon n$, $|S_2| \geq \varepsilon n$, then there is at least one edge joining a vertex of S_1 to a vertex of S_2.

Consider a random graph G on $(2+\varepsilon)n$ vertices and cn edges (where c will be specified later). There are $\binom{(2+\varepsilon)n}{n}$ ways that n vertices of G can be deleted. The remaining $n(1+\varepsilon)$ points can then be split into two sets S_1 and S_2 in at most $2^{n(1+\varepsilon)}$ ways. Thus, the total number of splittings is at most

$$\binom{(2+\varepsilon)n}{n} 2^{n(1+\varepsilon)} < 2^{(2+\varepsilon)n} 2^{n(1+\varepsilon)} < 2^{3(1+\varepsilon)n} .$$

Between S_1 and S_2 there are at least εn^2 potential edges. The probability that none of these edges actually occurs in G is less than $\left(1 - \frac{c}{(2+\varepsilon)n}\right)^{\varepsilon n^2}$. Thus, if c is chosen so that

$$(7) \qquad 2^{3(1+\varepsilon)n} \left(1 - \frac{c}{(2+\varepsilon)n}\right)^{\varepsilon n^2} \to 0$$

as $n \to \infty$ then we easily see that almost all graphs cannot be split in such a way.

Since

$$\left(1 - \frac{c}{(2+\varepsilon)n}\right)^{\varepsilon n^2} \to e^{-\left(\frac{\varepsilon c}{2+\varepsilon}\right)n}$$

then for c large enough, e.g., $c > 18(\varepsilon+\varepsilon^{-1})$,

$$e^{-\left(\frac{\varepsilon c}{2+\varepsilon}\right)n} < e^{-3(1+\varepsilon)n}$$

and (7) holds. This proves the theorem. ■

The other half of (6) is proved in a similar way. It would be interesting to determine the best possible value of c but this does not seem to be too easy.

We mention here the undirected analogue of (1). Let $g(n,n)$ denote the smallest integer for which there is an undirected graph of $g(n,n)$ edges so that if we omit any n of its vertices then there always remains a path of length n. We believe

$$\frac{g(n,n)}{n} \to \infty, \quad \frac{g(n,n)}{n \log n} \to 0$$

as $n \to \infty$ and hope to return to this question in finite time.

A related question is the following: Consider random graphs on n vertices and Cn edges. Is it true that for large C almost all of these graphs have a path of length $n(1-\varepsilon)$? It is known [4] that almost all graphs on n vertices and $(\frac{1}{2} + \varepsilon)\, n \log n$ edges are Hamiltonian.

It is possible to introduce another parameter into these questions. Let $F_v(t;n,n)$ denote the smallest integer for which there is a graph with t vertices and $F_v(t;n,n)$ edges having the property that if any n vertices are deleted there still remains a connected component with at least n vertices. If $t/n \to c > 2$ then $F_v(t;n,n)/n \to A(c)$ where $A(c) \to \infty$ as $c \to 2$. (The behavior of $F_e(t;n,n)/n$ is similar). We could also omit edges instead of vertices but leave the formulation of these questions to the reader.

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