ERROR BOUNDS IN THE APPROXIMATION OF EIGENVALUES OF DIFFERENTIAL AND INTEGRAL OPERATORS

by

F. Chatelin
J. Lemordant

STAN-CS-75-479
FEBRUARY 1975

COMPUTER SCIENCE DEPARTMENT
School of Humanities and Sciences
STANFORD UNIVERSITY

ERROR BOUNDS IN THE APPROXIMATION OF EIGENVALUES OF DIFFERENTIAL AND INTEGRAL OPERATORS

by

FRANÇOISE CHATELIN+ and JACQUES LEMORDANT++

February 1975

Abstract

Various methods of approximating the eigenvalues and invariant subspaces of nonself-adjoint differential and integral operators are unified in a general theory. Error bounds are given, from which most of the error bounds in the literature can be derived.

+ Stanford University and University of Grenoble (France).
Supported by IRIA, Rocquencourt, NSF Grant DCR-7203712 A02, and AEC Contract AT(04-3) 326 PA #30.

Part of this material was presented at the GATLINBURG VI meeting in Munich, December 15-22, 1974 (sponsored by the Sifterverband für die Deutsche Wissenschaft, through the German Research Council).

++ University of Grenoble

I. INTRODUCTION

We are concerned with the error bounds for the numerical computation of the eigenvalues of differential or integral operators.

T denotes a linear operator on a Banach space X , and T_n its approximation n = 1, 2,.... $\| . \|$ is the norm on the algebra L'(X) of bounded linear operators on X . I is the identity operator on X .

There exists a wide variety of approximation methods, the most important of which belong to one of the three following classes:

- Class 1 : uniform **approximation.**

Definition: $T, T_n \in \mathcal{L}(X)$, $\| T - T_n \| \to 0$

Example: the Rayleigh-Ritz and Galerkin methods, where the differential operator is approximated by restriction to a finite dimensional subspace. They correspond to the uniform approximation of the inverse, [2], [4], [6], [8] .

- Class 2 : **collectively** compact approximation.

Definition: $T, T_n \in \mathcal{L}(X)$, $(T - T_n) x \xrightarrow[n \to \infty]{} 0$ for any x in X , and the

sets $\{ (T-T_n) x ; \|x\| \leq 1 \}$, n = 1, 2,..., are relatively compact.

Remark T_n - T is T - compact, according to Kato ([5], p. 194).

Examples: (1) approximation of an integral operator by using approximate quadrature formulas (Anselone [1]). Consider X = C(0,1) with the uniform norm,

$T : f(x) \in X \mapsto \int_0^1 K(x,y)f(y)dy$, where K is continuous on $[0,1]^2$.

$T_n : f(x) \in X \mapsto \sum_{j=1}^{n} w_{n_j} K(x, y_{n_j}) f(y_{n_j})$, where $0 \leq y_{n_j} \leq 1$

.and the weights w_{n_j} such that : $\sum_{j=1}^{n} w_{n_j} f(y_{n_j}) \xrightarrow[n\to\infty]{} \int_0^1 f(x)\, dx$

for any f in X (the rectangular, trapezoidal, Simpson, Weddle and Gauss quadrature rules satisfy this condition).

(2) approximation of a differential operator by finite differences, when considering T^{-1} (Vainikko [9])

- Class 3 : neighboring approximation;

Definition: T, T_n are closed operators, with domain of definition $D(T) = D(T_n)$. $T - T_n$ is closed and $(T-T_n) x \xrightarrow[n\to\infty]{} 0$ for any x in D(T) .

$\| (T - T_n)(T-z1)^{-1} \| \xrightarrow[n\to\infty]{} 0$ for any z in C such that $(T-z1)^{-1} \in \mathcal{L}(X)$.

Examples: approximation of a differential operator by a neighboring differential operator (Pruess [7]) .

(1) Consider X = C (0,1) with the uniform norm,

$D = \{ x \in X : x'' \in X \text{ and } x(0) = x(1) = 0 \}$,

$T : x \in D \mapsto -x'' + q.x$,

$T_n : x \in D \mapsto -x'' + q_n.x$,

where q, q_n are real-valued continuous functions on [0,1]

and $\max_{0 \leq t \leq 1} | q(t) - q_n(t) | \xrightarrow[n\to\infty]{} 0$

$T-T_n$ is the multiplication operator defined by $q-q_n$,

$\| T - T_n \| = \| q-q_n \|_\infty \to 0$.

(2) A less obvious example is given by the following:

Consider $X = C(0,1)$ with the uniform norm ,

$$D = \{x \in X \; ; \; x'' \in X \text{ and } x(0) = x(1) = 0\} ,$$

$$T : x \in D \mapsto p_o u'' + p_1 u' + p_2 u ,$$

$$T_n : x \in D \mapsto p_o^{(n)} u'' + p_1^{(n)} u' + p_2^{(n)} u ,$$

where p_i , $p_i^{(n)}$, $i = 0,1,2$, are real valued continuous functions on $[0,1]$ and $\max_{0 \le t \le 1} | p_1 - p_1^{(n)}() | \to 0$. We suppose that $p_0 < 0$, $p_0^{(n)}() \le \delta <$.

$H_n = T - T_n$ is an unbounded operator, but it is T-bounded, according to Kato ([5], p. 189).

<u>Definition:</u> An operator A , whose domain D(A) includes D(T) is <u>T-bounded</u> if :

$$\| Ax \| \le a \| x \| + b \| Tx \| , \text{ for } x \text{ in } D(T) .$$

The proof that H_n is T-bounded is in [5], p. 193. We get

$$\| H_n x \| \le a_n \| x \| + b_n \| Tx \|, \text{ for } x \in D(T) , \text{ and } a_n, b_n \to 0 ,$$

Consider $x = R(z)y$, for z on Γ , enclosing an eigenvalue λ of T .

$$\| H_n R(z)y \| \le a_n \| R(z)y \| + b_n \| (T-z1)R(z)y + zR(z)y \|$$

$$\le [(a_n + |z| b_n) \| R(z) \| + b_n] \| y \| .$$

Thus $\| H_n R(z) \| \to 0$.

Various convergence proofs are given in the literature, adapted to each type of method under consideration: norm convergence for class 1 [8], compactness argument for class 2 [1], [9], norm convergence of the inverse for class 3 [5], (see [7] for the Sturm-Liouville operator).

We present here these three classes of approximation as special cases of a more general approximation. With this unifying treatment, we are able to give the general type of error bounds that hold for eigenvalues and the gap between invariant subspaces. It remains, however, for each special case, to derive specific error bounds from the general ones given here. It should be noted that the approximation theory proposed here applies to unbounded closed operators as well.

The approximation will be defined so that the Newmann series of the approximate resolvent is convergent. Then the approximate and exact invariant subspaces have the same dimension for n large enough and the approximate eigenvalues converge to the exact eigenvalue. The proofs depend heavily on the perturbation theory developed by Kato in [5]. The main results (theorems 1, 2, 3) are due to Jacques Lemordant (University of Grenoble).

II. THE APPROXIMATION T_n OF T

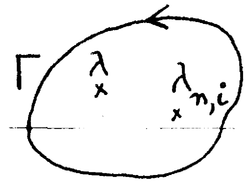
Let X be a Banach space, T a closed linear operator from x to x, with domain of definition D(T).

λ is an isolated eigenvalue of T, with finite algebraic multiplicity m.

Γ is a positively oriented rectifiable curve enclosing λ, but excluding any other point of the spectrum of T.

P is the spectral projection associated with λ :

$$P = \frac{-1}{2i\pi} \int_{\Gamma} (T - z1)^{-1} dz ,$$ PX is the invariant subspace associated with λ .



$R(z) = (T - z1)^{-1}$ is the resolvent of T,, for z in the resolvent set of T.

We want to approximate λ and PX.

Let T_n , n = 1, 2, ... be an approximation of T . The precise meaning of "approximation of T " is stated below: (2.1) to(2.4).

It will be shown in Section III that the spectrum of T_n inside Γ is discrete and that there are exactly m approximate eigenvalues for n large enough: $\lambda_{n,i}$, i = 1, ... m .

P_n is the spectral projection associated with all the eigenvalues of T_n lying inside Γ .

$R_n(z) = (T_n - z1)^{-1}$, for z in the **resolvent** set of T_n.

In general, we consider the approximation of λ by the arithmetic mean:

$$\lambda_n = \frac{1}{m} \sum_{i=1}^{m} \lambda_{n,i} .$$

λ_n is the weighted mean of the h-group, according to Kato [5] .

<u>Definition of the approximation</u> T_n .

Let T_n, n = 1,2, ... be a sequence of closed linear operators from x to X , with domain of definition $D(T_n)$, and such that:

(2.1) $D(T_n) \supset D(T)$, n = 1,2,...,

(2.2) $T - T_n$ is closed , n = 1,2,..., (2.3) $T_n x \underset{n\to\infty}{\longrightarrow} Tx$ for any x in $D(T)$,

(2.4) $\| [(T - T_n) R(z)]^2 \| \underset{n\to\infty}{\longrightarrow} 0$, for any z on Γ.

Then T_n is said to be an <u>approximation of T</u> .

First we need the following:

<u>Lemma 1</u> | $(T-T_n)R(z)$ is uniformly bounded in n , for any z on Γ , and $\| (T - T_n) P \| \underset{n\to\infty}{\to} 0$.

<u>Proof</u> : Since $T - T_n$ is closed, and $R(z)$ is a bounded operator on X with range $D(T)$, $(T - T_n)R(z)$ is a closed operator with domain X , hence bounded for any n , by the closed graph theorem.

$(T-T_n)R(z)x \to 0$ for any x in X , then $(T - T_n)R(z)$ is uniformly bounded in n by the principle of uniform boundedness. On the other hand, $(T - T_n)P$, which converges pointwise to zero, converges uniformly on the finite dimensional **subspace** PX .

Let S be the reduced resolvent in $z = \lambda$, S =&ii R(z)(1-P) .

<u>Lemma 2</u> | $\|((T - T_n)R(z))^2 \| \to 0$ implies $\|((T - T_n)S)^2 \| \to 0$.

<u>Proof</u> : Let $H_n = T - T_n$.

$$H_nR(z)(1 - P) H_nR(z)(1 - P) = (H_nR(z))^2 - H_nR(z)PH_nR(z) - H_nR(z)H_nR(z)P + H_nR(z)PH_nR(z)P .$$

Since T and P commute, $R(z)P = PR(z)$. Then:

$$\|(H_nR(z)(1-P))^2\| \leq \|(H_nR(z))^2\| + \|H_nP\| \, \|R(z)\| \, \|H_nR(z)\| (2 + \|P\|)$$

$$\xrightarrow[n \to \infty]{} 0 \text{ , for any } z \text{ on } \Gamma \text{ .}$$

Since $(H_nR(z)(1-P))^2$ is holomorphic in z inside Γ, its norm at $z = \lambda$ is less than or equal to its norm at any point z on Γ. We then have: $\|(H_nS)^2\| \to 0$.

Remark: $\|(H_nR(z))^2\| \to 0$ for z on Γ implies that it tends to zero for any $z \neq \lambda$ inside Γ, as it is easily shown: $(H_nR(z))^2 = (H_nR(z)(P + 1 - P))^2$ can be expressed in terms of $H_nR(z)P = H_nPR(z)$ and $H_nR(z)(1 - P)$ which is holomorphic inside Γ. The desired result follows.

The definition of T_n includes the three classes defined above:

Class 1 : T, T_n bounded and $\|T - T_n\| \to 0$.

Class 2 : T, T_n bounded and $((T - T_n)B\}$ relatively compact where B is the unit ball of X .

Then $\Sigma = \{ T - T_n) R(z)B \}$ is relatively compact for any z on Γ and $(T - T_n)R(z)$, which is bounded on X and converges pointwise to zero, converges uniformly on Σ , i.e. (2.4) .

Class 3: T, T_n, $T - T_n$ closed and $\|(T - T_n)R(z)\| \to 0$, for z on Γ .

III. EXISTENCE OF THE SECOND NEUMANN SERIES OF $R_n(z)$

Let H_n denote $T - T_n$: $T_n = T - H_n$ and let z be any point on Γ .

<u>The key point in the whole theory is the following:</u>

<u>Lemma3</u> $R_n(z)$ can be represented by the second Neumann series:

$$R_n(z) = R(z) \sum_{k=0}^{\infty} (H_n R(z))^k$$

<u>Proof:</u> (3.1) $T_n - z1 = T - z1 - H_n = (1 - H_n R(z)) \; (T - z1)$.

$(1 - H_n R(z))^{-1}$ exists and is represented by $\sum_{k=0}^{\infty} (H_n R(z))^k$, if this series is convergent.

$$\sum_{k=0}^{\infty} (H_n R(z))^k = (1 + H_n R(z)) \sum_{k=0}^{\infty} (H_n R(z))^{2k} ,$$

and by (2.4), $\sum_{k=0}^{\infty} H_n R(z))^{2k}$ is a convergent series for n large enough. Then, from (3.1), we get the expansion of the lemma.

<u>Remarks</u> (1) $R_n(z) - R(z) = R(z) \sum_{k=1}^{\infty} (H_n R(z))^k$

$$(3.2) \qquad = R(z) H_n R(z) + R(z) (1 + H_n R(z)) \sum_{k=1}^{\infty} (H_n R(z))^{2k} .$$

Put $\varepsilon_n = \max_{z \in \Gamma} \| (H_n R(z))^2 \|$, $\| \sum_{1}^{\infty} (H_n R(z))^{2k} \| \leq \frac{\varepsilon_n}{1-\varepsilon_n}$.

In general, $R_n(z)$ does not converge to $R(z)$ in norm. But it does, for example, for T_n in class 1 ($\|H_n\| \to 0$) or in class 3 ($\|H_n R(z)\| \to 0$). So,- if T_n is in class 3, $(T_n - z1)^{-1}$ is an approximation of $(T-z1)$ which belongs to class 1 .

(2) Lemma 3 would be still valid if the assumption (2.4) was replaced by: $\exists\, p>0$ such that $\|(H_n R(z))^p\| \to 0$ for z on Γ.

Corollary 1 | There are exactly m eigenvalues of T_n converging to λ when n tends to infinity.

Proof : Let n be fixed such that $\|(H_n R(z))^p\| < 1$. And consider the perturbation of T defined by:

$$x \in [0,1] \; : \; T(x) = T - xH_n .$$

$T(0) = T$ and $T(1) = T_n$. The second Neumann series of $(T(x) - z1)^{-1} = R(x,z)$ exists for any x in $[0,1]$.

When $x \to 0$, $\| R(x,z) - R(z) \| \to 0$ and $\| P(x) - P \| \to 0$.

For x small enough such that $\| P(x) - P \| < 1$, $\dim P(x)X = m$.

But $P(x)$ is uniformly continuous in x on $[0,1]$, we then deduce that $\dim P(1)X = m$.

This means that there are exactly m eigenvalues of T_n inside Γ.

Since this is true for any curve Γ' inside Γ, arbitrarily close to λ (because (2.4) holds for any $z \neq \lambda$ inside Γ), then:

$$\lim_{n\to\infty} \lambda_{n,i} = \lambda , \; i = 1, \ldots, m.$$

T_n is said to be a strongly stable approximation of T (Chatelin [3]).

$$(P_n - P)x = \frac{-1}{2i\pi} \int_\Gamma (R_n(z) - R(z))x \, dz \; , \text{ for any } x \in X .$$

From (3.2) we get readily that $\|(P_n - P)x\| \xrightarrow[n\to\infty]{} 0$.

Since PX is m-dimensional, we even get $\|(P_n - P)P\| \to 0$.

Following [1] and [8] :

$$(P_n - P)Px = \frac{-1}{2i\pi} \int_\Gamma (R_n(z) - R(z))Px\, dz \text{, for any } x \text{ in } X ,$$

$$= \frac{-1}{2i\pi} \int_\Gamma R_n(z)\,(T_n - T)\, R(z)Px\, dz .$$

$R(z)P = PR(z)$, then:

$$\|(P_n - P)P\| \leq \frac{m(\Gamma)}{2\pi} \max_{z\in\Gamma}\left(\|R_n(z)\|\,\|R(z)\|\right) . \|(T-T_n)P\| ,$$

where $m(r)$ is the length of Γ, and $\max_{z\in\Gamma} \|R_n(z)\|$ is uniformly bounded in n . Since the dimensions of PX and P_nX are the same for n large enough, it is not difficult to carry out a bound for the gap between PX and P_nX (see definition in Section V) in terms of $\|H_nP\|$.

E'or the eigenvalues, a bound of type: $|\lambda_n - \lambda| \leq K \|H_nP\|$ can be derived, in this general setting, following the lines of the proof given in [8] for a collectively compact approximation.

In order to get a more precise expression for the bound, we have to go into a more detailed analysis of the perturbation of T by $T_n - T = -H_n$.

IV. THE OPERATOR $P_n - P$.

<u>Theorem 1</u> | There exists a decomposition : $_nP - P = P_{1n} + P_{2n}$,

such that : a) $P_{1n} \in \mathcal{L}(X)$, $P_{1n}X \subset PX$, $P_{1n}P = 0$,

b) $P_{2n} \in \mathcal{L}(X)$, $\|P_{2n}\| \leq K \|H_nP\|$,

for n large enough.

The proof of theorem 1 contains five intermediate steps.

Proof :

1. λ , an eigenvalue of finite algebraic multiplicity m , is a pole of order ℓ $(1 \leq \ell \leq m)$ of R(z) , whose Laurent expansion can be written ([5], p. 180) :

$$R(z) = \sum_{k=-\ell}^{\infty} (z-\lambda)^k S^{(k+1)},$$

with $S^{(o)} = -P$

$$S^{(-k)} = -D^k, \quad k \geq 1, \quad D = (T - \lambda 1)P$$

$$S^{(k)} = S^k, \quad k \geq 1, \quad S = \lim_{z \to \lambda} R(z)(1-P)$$

Using the Neumann series of $R_n(z)$, we get:

$$P_n - P = \frac{-1}{2i\pi} \int_\Gamma R(z) \sum_{i=1}^{\infty} (H_n R(z))^i \, dz$$

$$= \sum_{i=1}^{\infty} P_{n,i},$$

with: $P_{n,i} = \dfrac{-1}{2i\pi} \displaystyle\int_\Gamma R(z)(H_n R(z))^i \, dz$

$$= \frac{-1}{2i\pi} \int_\Gamma R(z) \left(H_n \sum_{k=-\ell}^{\infty} (z-\lambda)^k S^{(k+1)}\right)^i dz .$$

$$(4.1) \quad P_{n,i} = \sum_{\substack{k_1+k_2+\ldots+k_{i+1} = i \\ k_j \geq -\ell+1,\ j=1,\ldots,i+1}} S^{(k_1)} \underbrace{H_n S^{(k_2)} \ldots S^{(k_i)} H_n S^{(k_{i+1})}}_{U_n^{(k_j)}},$$

(cf Kato [5], p.76).

2. It is easy to show that a'theorem 1-type decomposition $P'_{1n} + P'_{2n}$ holds with $\| P'_{2n} \| \to 0$:

Let us go back to the expansion (3.2). By integration on Γ :

$$P_n - P = \frac{-1}{2i\pi} \left[\int_\Gamma R(z) H_n R(z)\, dz + \int_\Gamma R(z)\,(1 + H_n R(z))\left[\sum_1^\infty (H_n R(z))^{2k}\right] dz \right]$$

If we substitute in the first integrand the Laurent expansion of $R(z)$, only the coefficient of $\frac{1}{z-\lambda}$ contributes to the integral :

$$\frac{-1}{2i\pi} \int_\Gamma R(z)\, H_n R(z)\, dz = S^\ell H_n D^{\ell-1} + S^{\ell-1} H_n D^{\ell-2} + \ldots + S H_n P$$
$$- P H_n S - D H_n S^2 - \ldots - D^{\ell-1} H_n S^\ell .$$

obviously : $\| S^\ell H_n D^{\ell-1} + \ldots + S H_n P \| \leq K \| H_n P \|$,

and $P'_{1n} = -P H_n S - \ldots - D^{\ell-1} H_n S^\ell$ has its range included in PX, and $P'_{1n} P = 0$.

The second integral can be bounded in norm by :

$$m(\Gamma) \max_{z\in\Gamma} \| R(z) \| \cdot (1 + \| H_n R(z) \|) \frac{\varepsilon_n}{1-\varepsilon_n} ,$$

where $m(\Gamma)$ is the length of Γ. Then : $\| P'_{2n} \| \to 0$.

In order to bound $\| P_{2n} \|$ in terms of $\| H_n P \|$, we have to go back to the expansion of $P_n - P$ in terms of $P_{n,i}$.

3. Consider (4.1).

Let $N(i)$ be the number of terms in that sum. $N(i)$ is also the absolute value of the coefficient of z^i in the series expansion of

$$\left[\sum_{k=-\ell+1}^{\infty} z^k \right]^{i+1} ,$$

or else the coefficient of $z^{\ell i + \ell - 1}$ in the expansion of

$$\left[\sum_{k=0}^{\infty} z^k \right]^{i+1} = (1 - z)^{-i-1} .$$

$$\frac{d^i}{dz^i}\,\frac{1}{1-z} = \frac{(-1)^i i!}{(1-z)^{i+1}} = \frac{d^i}{dz^i}(1 + z + z^2 + \dots) .$$

The coefficient of z^s in $\frac{1}{(1-z)^{i+1}}$ is then $\frac{(i+s)(i+s-1)\dots(s+1)}{i\,!} = C^s_{i+s}$

$$N(i) = C^{\ell i+\ell-1}_{(\ell+1)i+\ell-1} .$$

Making use of the Stirling formulae, we can easily show that there exists a constant a (depending on ℓ only) such that:

$$N(i) \leq a^i , \quad i = 1, 2, \dots .$$

4. P_{1n} will be the sum of all $U_n^{(k_j)}$ whose norm is not going to zero (such that $S^{(o)}H_nS$, $S^{(-2)}H_nS^3H_nS$, etc..). Such terms correspond to sequences $(k_1, k_2, \dots, k_{i+1})$ $i = 1, 2, \dots,$ in which $k_j \geq 1$ for $j \geq 2$, since $\| H_nS^{(k)} \| \to 0$ for any nonpositive k ,

$k_2 + \dots + k_{i+1} \geq i$ implies $k_1 = i - (k_2 + \dots + k_{i+1}) \leq 0$. Then each operator such that $k_1 \leq 0$, $k_j \geq 1$ for $j \geq 2$ is a bounded operator with range in PX .

We have to prove that P_{1n} is bounded.

$$P_{1n} = \sum_{i=1}^{\infty} \Big[\sum_{\substack{k_1+k_2+\dots k_{i+1} \\ -\ell+1 \leq k_1 \leq 0 \\ k_j \geq 1 ,\ j=2,\dots ,\ i+1}} S^{(k_1)} H_n S^{(k_2)} \dots H_n S^{(k_{i+1})} \Big|$$

Let us **recall** that $\eta_n = \|(H_nS)^2\| \underset{n\to\infty}{\longrightarrow} 0$. We shall prove that for i large enough, each $U_n^{(k_j)}$ in the above sum contains enough factors of the $(H_nS)^2$ type, in order to ensure the absolute convergence of $\sum_{i=1}^{\infty} [S_i]$.

Namely:

For $i \geq 2\ell - 1$, each $U_n^{(k_j)}$ with $k_1 \leq 0$, $k_j \geq 1$ for $j \geq 2$, contains at least $\mathcal{P}e\left(\frac{i-21+3}{2}\right)$ times the factor $(H_n S)^2$, and at most $\ell - 1$ times the factor $H_n S^k$, $k = I, .., a$, where $\mathcal{P}e\,(x)$ is the integer part of x.

This is shown by a close study of the sequence of exponents k_j subjected to the above constraints.

Then, for $i \geq 2\ell - 1$:

$$\| S_i \| \leq a^i \eta_n^{\mathcal{P}e\left(\frac{i-2\ell+3}{2}\right)} K_1^{\ell-1} K_2 \,,$$

where $K_1 = \max\limits_{k=?,..,\ell} \sup\limits_n \| H_n S^k \|$, $K_2 = \max\limits_{k=0,..,\ell-1} \| D^k \|$.

The series P_{1n} will be absolutely convergent for n large enough so that $a\eta_n^{\frac{1}{2}} < 1$.

$P_{In} P = 0$ follows from $SP = 0$.

5. P_{2n} will be the sum of all $U_n^{(k_j)}$ for which there exists a $j \in \{2, .., i+1\}$ such that $k_j \leq 0$. Let us then decompose P_{2n} into:

$$P_{2n} = \sum_{p=1}^{\infty} \sum_{i=1}^{\infty} \sum_{*} U_n^{(k_j)} = \sum_p \sum_i \sigma_{p,i}$$

$$* \begin{cases} k_1 + k_2 + \ldots k_{i+1} = i \\ k_j \geq -\ell + 1 \,,\ i = 1, 2, .., i+1 \\ \text{there exist exactly p indices } j\,,\ j \in \{2, ..., i+1\} \\ \text{such that } k_j \leq 0\,. \end{cases}$$

Consider $\sigma_{p,i}$ for a given i and p .

$$\sum_{j\neq 1, k_j \geq 1} k_j \leq i + (p+1)(R-1) .$$

For $i \geq (p+1)(2\ell+1)$, each $U_n^{(k_j)}$, with k_j satisfying the constraints * , contains at least $pe\left(\frac{i-(p+1)(2\ell+1)+2}{2}\right)$ times the factor $(H_n S)^2$, at most $(p+1)\ell-1$ times the factor $H_n S^k$, $k = 1,\ldots,(p+1)(2\ell-1)$, and p factors of the type $H_n D^k$, $k = 0,1,\ldots,\ell-1$.

$$\| H_n S^k \| \leq \| H_n S \| \, \| S^{k-1} \| < \| H_n S \| \, \| S \|^{k-1}$$

$H_n D^k = H_n (T-\lambda 1)^k P = H_n P (T-\lambda 1)^k P$, since P and T commute.

Then $\| H_n D^k \| \leq \| H_n P \| \, \| D^k \|$, $k = 0,\ldots,\ell-1$.

We get , for $i \geq (p+1)(2\ell+1) = I(p)$:

$$\| \sigma_{p,i} \| \leq \left[a^i \, \eta_n^{pe\left(\frac{i-(p+1)(2\ell+1)+2}{2}\right)} . \| H_n P \|^p \, K_2^p \, \| H_n S \|^{(p+1)(2\ell-1)} \right.$$

$$\left. \| S \|^{(p+1)^2 (2\ell-1)\ell} \right] .$$

$$\| P_{2n} \| \leq \sum_{p=1}^{\infty} K_3^p \, \| H_n P \|^p \left(\sum_{i=1}^{I-1} a^i + \sum_{i=I}^{\infty} a^i \eta_n^{\frac{1}{2}} \right) .$$

$$\sum_{i=1}^{I-7} a^i = a \frac{a^{I-1} - 7}{a-7} < \frac{a^I}{a-1} .$$

$$\sum_{I}^{\infty} \left(a\eta_n^{\frac{1}{2}} \right)^i < \left(a\eta_n^{\frac{1}{2}} \right)^I \frac{1}{1-a\eta_n^{\frac{1}{2}}} \text{ , for } n \text{ such that } a\eta_n^{\frac{1}{2}} < 7 .$$

Hence $\| P_{2n} \| \leq \sum_{p=1}^{\infty} K_4^{\,p} \| H_n P \|^p$

$$\leq K_5 \| H_n P \| ,$$

which completes the proof of theorem 1 .

Corollary 2 | For n large enough :

$$\| (P_n - P) P \| \leq K \| H_n P \| ,$$

$$\| (P_n - P) P_n \| \leq K \| H_n P \| .$$

Proof : $(P_{r-i-} P) P = P_{2n} P$,

$$(P_n - P) P_n = (P_n - P)^2 + (P_n - P) P ,$$

and $(P_n - P)^2 = P_{2n} P_{1n} + P_{1n} P_{2n} + P_{2n}^2$, since $P_{1n} P = 0$

The results then follow.

For approximations of class 1 and 3, we have : $\| P - P_n \| \to 0$.

V. CONVERGENCE IN GAP OF THE INVARIANT SUBSPACES

Let us borrow from Kato ([5], p.197), the definition of the gap between two closed subspaces M and N , of a Banach space X :

$$\delta (M,N) = \sup_{\substack{x \in M \\ \|x\| = 1}} \text{dist} (x,N) ,$$

$\hat{\delta} (M,N) = \max [\delta(M,N), \delta(N,M)]$ is the gap between M and N .

The following property holds: $\delta(M,N) < 1$ implies $\dim M \leq \dim N$, and $\hat{\delta} (M,N) < 1$ implies $\dim M = \dim N$.

Theorem 2 — For n large enough:

$$\hat{\delta}(PX, P_nX) \leq K \| H_nP \| .$$

Proof : We have the inequalities :

$$\delta(PX, P_nX) \leq \|(P_n - P)P\|,$$

and $\delta(P_nX, PX) \leq \|(P_n - P)P_n\|$.

Theorem 2 follows from Corollary 2.

Remark : If T_n is an approximation of T such that $R_n(z)x \to R(z)x$ for any $x \in X$, and any z on Γ , then $P_nx \to Px$ and, since PX is finite-dimensional, $\|(P_n - P)P\| \to 0$. This implies that dim PX $\leq$ dim P_nX : there are at least m approximate eigenvalues lying inside Γ . We need some additional assumption to show that $\|(P_n - P)P_n\| \to 0$ and dim $P_nX \leq$ dim PX . This assumption is provided here by the hypothesis (2.4).

VI. CONVERGENCE OF THE EIGENVALUES

6.1 Series expansion of $\lambda_n - \lambda$

The trace of a linear operator A with finite rank is denoted by tr A. If A is of finite rank and B continuous, the identity tr AB = tr B A holds, (Kato [5] p. 379).

For the following, refer to Kato [5], p. 77.

$$\mathrm{tr}\ T_nP_n = \sum_{i=1}^{m} \lambda_{n,i} ,$$

$$(T_n - \lambda 1) R_n(z) = 1 + (z - \lambda) R_n(z) .$$

$$(T_n - \lambda 1) P_n = \frac{-1}{2i\pi} \int_\Gamma (T_n - \lambda 1) R_n(z)\, dz$$

$$= \frac{-1}{2i\pi} \int_\Gamma (z - \lambda) R_n(z)\, dz = \frac{-7}{2i\pi} \int_\Gamma (z-\lambda) R(z) \sum_{p=0}^{\infty} (H_n R(z))^p\, dz$$

$$= (T-\lambda 1)\, P - \frac{dz}{2i\pi} \int_\Gamma (z-\lambda) R(z) \sum_{p=1}^{\infty} (H_n R(z))^p$$

$$\lambda_n - A = \frac{1}{m} \operatorname{tr} (T_n - \lambda 1) P_n = \frac{-1}{2i\pi m} \operatorname{tr} \int_\Gamma (z-\lambda) R(z) \sum_{p=1}^{\infty} (H_n R(z))^p dz ,$$

since $\operatorname{tr} (T-11)\, P = 0$.

Using $\frac{d\, R(z)}{dz} = (R(z))^2$, we get

$$\frac{d}{dz} (H_n R(z))^p = \frac{d}{dz} [H_n R(z) \ldots H_n R(z)] = H_n R(z) \ldots H_n R^2(z) + \ldots + H_n R^2(z) \ldots H_n R(z).$$

$$\operatorname{tr} \int_\Gamma (z-h) \frac{d}{dz} (H_n R(z))^p\, dz = p \operatorname{tr} \int_\Gamma (z-\lambda)(H_n R(z))^p\, dz .$$

This can be proved by using the Laurent expansion in λ of $R(z)$, integrating on Γ , then using tr AB = tr BA , since each term contains P at least once. Then :

$$\lambda_n - A = \frac{-1}{2i\pi m} \sum_{p=1}^{\infty} \operatorname{tr} \int_\Gamma \frac{1}{p} (z-\lambda) \frac{d}{dz} (H_n R(z))^p\, dz$$

$$= \frac{1}{2i\pi m} \sum_{p=1}^{\infty} \operatorname{tr} \int_\Gamma \frac{1}{p} (H_n R(z))^p\, dz \quad \text{(integration by parts)}$$

$$(6.1) \quad \lambda_n - A = \frac{1}{m} \sum_{p=1}^{\infty} \frac{1}{p} \operatorname{tr} \sum_{\substack{k_1+k_2+\ldots k_p = p-1 \\ k_j \geq -\ell+1,\ j=1,\ldots,p}} H_n S^{(k_1)} \ldots H_n S^{(k_p)}$$

6.2. We prove the following:

Theorem 3 | For n large enough:

$$| \lambda_n - \lambda | \leq \frac{1}{m} | \mathrm{tr}\, H_n P | + K \| H_n P \| .$$

Proof : All operators which appear in (6.1) contain at least one operator with finite rank, so we can apply the bound :

$$\frac{1}{m} | \mathrm{tr}\, A | \leq \| A \| .$$

For p = 1 we get $\frac{1}{m} \mathrm{tr}\, H_n P$, which appears to be the principal term in $\lambda_n - \lambda$ for most approximation methods.

$\sigma = \frac{1}{m} \sum_{p=2}^{\infty} \frac{7}{p} \mathrm{tr} \sum H_n S^{(k_1)} \ldots H_n S^{(k_p)}$ can be easily bounded in norm by $K \| H_n P \|$ by using the technique developed in Section IV .

Corollary 3 | For n large enough:

$$|\lambda_n - \lambda| \leq \frac{1}{m} \left| \mathrm{tr}\, H_n P + \sum_{p=2}^{\infty} \mathrm{tr} \sum_{\substack{k_1 + \ldots + k_p = p-1 \\ k_j \geq 1,\ j=1,\ldots,p-1 \\ -\ell+1 \leq k_p \leq 0}} H_n S^{(k_1)} \ldots H_n S^{(k_p)} \right| + K \| H_n P \|^2$$

Proof : As previously, we can decompose the sum over the k_j into the sum over the k_j where one k_j is nonpositive, then two k_j are nonpositive, and so on. The result above is obtained by considering one $k_j \leq 0$, and noticing that we have p operators with the same trace.

For example, if λ is a semi-simple eigenvalue, $\ell=1$, $k_p=0$, $k_1+..+k_{p-1} = p-1$ implies $k_j=1$, $j=1,..,p-1$, so that the sum. $\sum_2^\infty$ reduces to: $-tr(H_nSH_nP + (H_nS)^2 \sum_{p=o}^{\infty} (H_nS)^p \, . \, H_nP)$.

VII. APPLICATIONS

7.1. uniform approximation

$\| H_n \| \to 0$ implies $\| H_n^* \| \to 0$, where H_n^* is the **adjoint** of H_n .

We can then bound σ more precisely.

Theorem 4 For n large enough:

$$| \lambda_n - \lambda | < \frac{1}{m} | \operatorname{tr} H_nP | + K \| H_nP \| \; \| H_n^*P^* \| ,$$

$$| \lambda_n - \lambda | \leq \frac{1}{m} | \operatorname{tr} (H_nP - \sum_{k=1}^{\ell} H_nS^k H_nD^{k-1}) | + K \| H_n \| \; \| H_nP \| \, \| H_n^*P^* \| .$$

Proof: Consider σ :

For p = 2 we get : $-\frac{1}{m} \operatorname{tr} (H_nSH_nP + H_nS^2 H_nD \ . \ . \ . + H_nS^{\ell}H_nD^{\ell-1})$. For $1 \leq k \leq \ell$, $\operatorname{tr} H_nS^kH_nD^{k-1} = \operatorname{tr} PH_nS^kH_nPD^{k-1}$, then :

$$\left| \frac{1}{m} \operatorname{tr} \left(\sum_{k=1}^{\ell} H_nS^kH_nD^{k-1} \right) \right| \leq K \; \| H_nP \| \, \| H_n^*P^* \| .$$

For p = 3 , the bound is given by : $K^2 \|H_n\| \, \| H_nP \| \; \| H_n^*P^* \|$.

then: $\| \sigma \| \leq \left(\sum_{p=2}^{\infty} K^{p-1} \| H_n \|^{p-2} \right) \|H_nP\| \; \| H_n^*P^* \| \leq K \|H_nP\| \, \|H_n^*P^*\|$.

The second bound then follows.

For example, if $\ell = 1$, we get the principal term :

$$\frac{1}{m} \left| \operatorname{tr} (1 - H_n S) H_n P \right| .$$

Remark : The first bound is the same as the one given by Osborn [8] :

$\frac{1}{m} \operatorname{tr} H_n P = \sum_{j=1}^{m} ((T-T_n) \varphi_j, \varphi_j^*)$, where $(\varphi_j)_{j=1,..,m}$ is a basis of PX and $(\varphi_j^*)_{j=1,..,m}$ the adjoint basis of P*X* . On the other hand, $\|(T-T_n)P\| \leq \|P\| . \|(T-T_n)|_{PX}\|$. See [8], [2] for various examples.

Using the second bound we can derive the asymptotic equalities that we get in [4] , for a Galerkin-type approximation of a normal operator in a Hilbert space if $T_n = \pi_n T \pi_n$, where π_n is a sequence of orthogonal projections such that $\pi_n x \to x$, $x \in X$, then:

$$\frac{7}{m} \operatorname{tr} (\pi_n T - T)P = \lambda \sum_{j=1}^{m} ((1-\pi_n)\varphi_j, \varphi_j) = \lambda \|(1-\pi_n)\varphi\|^2$$

where cp belongs to PX .

7.2. collectively compact approximation

Obviously the bound in theorem 3 holds. It has to be compared to the bound : $|\lambda - \lambda_n| \leq K \, \mathrm{I}|(H_n)|_{PX}\|$ given by Osborn [8] .

Theorem 5 For n large enough:

$$|\lambda_n - \lambda| < \frac{L}{m} \left| \operatorname{tr} H_n P - \operatorname{tr} \sum_{k=1}^{\ell} H_n S^k H_n D^{k-1} \right| + \alpha_n \|H_n P\| ,$$

where $\alpha_n \xrightarrow{n \to \infty} 0$

Proof: Consider $\bar{\sigma} = \frac{1}{m} \sum_{p=3}^{\infty} \operatorname{tr} \sum_{\substack{k_1+\dots+k_p=p-1 \\ k_j\geq 1,\ j=1,\dots,p-1 \\ -\ell+1\leq k_p\leq 0}} H_n S^{(k_1)} \dots H_n S^{(k_p)}$.

Since T_n is collectively compact , $\| H_n S^r H_n S^t \| \to 0$, for $r, t \geq 1$.

Let $\varepsilon_n = \max_{(r,t)\in V} \| H_n S^r H_n S^t \|$, where V is the finite set of indices:

$$V = \{ (1,\ell) ,\ (1,\ell-1) , (2,\ell-1) , \ \vdots \ (1,1) \ (2,1) \ \dots \ (\ell, 1) \} .$$

$\| \bar{\sigma} \| \leq \sum_{p=3}^{\infty} a^p \ \varepsilon_n^{\operatorname{Pe}(\frac{p-1}{2})} \leq K \varepsilon_n$ for n large enough.

Theorem 5 follows from corollary 2, with $\alpha_n = K \varepsilon_n + \| H_n P\|$.

7.3. T_n belongs to class 3

Since $\| H_n R(z) \| \to 0$, for $z\in\Gamma$, $\| H_n S \| \to 0$.

Theorem 6 | For n large enough :

$$| \lambda_n - \lambda | \leq \frac{1}{m} | \operatorname{tr} H_n P | + \alpha_n \| H_n P \|$$

Proof : This follows readily from theorem 5 and $\| H_n S \| \to 0$.

If T and T_n are self- adjoint in a Hilbert space we get the bounds for n large enough:

$$| \lambda_n - \lambda | \leq \frac{1}{m} | \mathrm{tr}\, H_n P | + K \| H_n P \|^2 ,$$

$$| \lambda_n - a | \leq \frac{1}{m} | \mathrm{tr}\, (H_n P - \sum_{k=1}^{\ell} H_n S^k H_n D^{k-1}) | + K \| H_n S \| \| H_n P \|^2 .$$

The proof is easily adapted from the proof of theoren 4 by using the fact that $\| H_n S \| \to 0$.

7.4. T has a compact resolvent

Since $R(z)$ is compact, $\| H_n S^k \| = \| H_n S . S^{k-1} \| \to 0$ for $2 \leq k \leq \ell$.

Theorem 7 For n large enough :

$$| \lambda_n - \lambda | < \frac{1}{m} | \mathrm{tr}\, ((1 - H_n S) H_n P) | + \alpha_n \| H_n P \|$$

F. C.

Computer Science Department
Numerical Analysis
Stanford University
Stanford, California 94305

J. L.

I.R.M.A. Mathématiques Appliquees
BP 53
38041 Grenoble-Cedex
France

References

1. Anselone, P. M.,"Collectively compact operator approximation theory," Prentice Hall (1971).

2. Bramble, J. H., Osborn, J. E.,"Rate of convergence estimates for nonself-adjoint eigenvalue problems," Math. Comp., Vol.27, No.123, pp. 525-549, (1973).

3. Chatelin, F.,"Convergence of approximation methods to compute eigen-elements of linear operators," SIAM J. Numer. Anal. Vol.10, No.5, pp. 939-948, (1973).

4. Chatelin, F., Lemordant, J., "La méthode de Rayleigh-Ritz appliquée à des opérateurs différentiels--ordre de convergence des elements propres," to appear in Numer. -Math.

5. Kato, T., 'Perturbation theory for linear operators,' Springer Verlag, Berlin, (1966).

6. Lemordant J., "Problème quadratique aux valeurs propres et approximation," CRAS t.277, Ser. A,pp. 857-860.

7. Pruess, S., 'Estimating the eigenvalues of Sturm-Liouville problems by approximating the differential equation." SIAM J. Numer. Anal., Vol.10, No.1, pp. 55-68, (1973).

8. Osborn, J. E., "Spectral approximation for compact operators," Technical Report 74-26, University of Maryland, College Park, (1974).

9. Vainikko, G. M., "The compact approximation principle in the theory of approximation methods," Zh. Vychisl. Mat. Fiz.9, pp. 739-761 (1969).