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A NOTE ON A COMBINATORIAL PROBLEM OF BURNETT AND COFFMAN

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A problem arising in the analysis of interleaved memories is shown to be identical to a well-known problem in the combinatorial literature. The former problem concerns the number of sequences of length k drawn from the integers (1,2...n) such that each sequence contains distinct integers and does not contain a subsequence of the form (...i,i+I,....). The corresponding conbinatorial problem concerns derangements, that is, the class of permutations in which no element is left invariant by permutation. In the interleaved memory problem when k=n, the number or sequences is n!/e, which is the same as the number of derangements on n letters.

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A note on a combinatorial problem of Burnett and Coffman

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ABSTRACT

A problem arising in the analysis of interleaved memories is shown to be identical to a well-known problem in the combinatorial literature. The former problem concerns the number of sequences of length k drawn from the integers $\{1,2,\ldots,n\}$ such that each sequence contains distinct integers and does not contain a subsequence of the form $(\ldots i, i+1, \ldots)$. The corresponding combinatorial problem concerns derangements, that is, the class of permutations in which no element is left invariant by the permutation. In the interleaved memory problem, when $k=n$, the number of sequences is $n!/e$, which is the same as the number of derangements on n letters.

I. Introduction

Burnett and Coffman [1973] treat a problem that arises in the analysis of interleaved memories. The problem is to determine $C_{n,k}$ where $C_{n,k}$ is the number of sequences of length k drawn from the set of integers $\{1,2,...,n\}$ such that

(i) each sequence has k distinct integers;

(ii) the initial integer of each sequence is 1; and

(iii) no sequence contains the subsequence $(...,i,i+1,...)$.

The third property states that each sequence counted by $C_{n,k}$ has no successor transitions. In the Burnett-Coffman problem each sequence represents a collection of k distinct memories that are the targets of k distinct address references. The reason for the restriction on successor transitions is due to the Markov process that they assume to generate the address references. They show that the entire analysis depends only on the sequences counted by $C_{n,k}$. Note that the successor of memory module n is memory module 1, so that the transition $(...,n,1,...)$ is a successor transition. However, by restricting our attention to sequences that begin with a 1, we need never treat transitions of the form $(...,n,1,...)$, and we enumerate precisely $1/n^{th}$ of the sequences of interest.

The central point of this note is that the Burnett-Coffman problem is isomorphic to the well-known combinatorial problem of derangements. [cf. Liu, 1968]. A derangement of n letters is a permutation on n letters in which no letter is mapped onto itself. We show that $C_{n,n}$ is equal to the number of derangements on n-1 letters. More generally we define a k-derangement on n letters to be a mapping from the set $\{1,2,...,k\}$ onto the set $\{1,2,...,n\}$ such that the k images are distinct, and no element is mapped back onto itself. Then there is one-to-one correspondence between the k-1-derangements on n-1 letters and the sequences counted by $C_{n,k}$.

There are various ways of establishing the one-to-one correspondence. We might proceed by finding a one-to-one correspondence between the $C_{n,k}$ sequences and k-1-derangements on n-1 letters, but this is rather tedious, even though many such maps exist. Since the computation of the number of k-derangements on n letters is very simple, we proceed by applying the derangement counting technique to the Burnett-Coffman problem and establish the correspondence by showing that the solutions are identical.

II. The derivation of $C_{n,k}$

The calculation of $C_{n,k}$ uses an inclusion-exclusion argument. In this discussion we use the notation $(n)_k$ to denote the falling factorial $n(n-1)(n-2)\ldots(n-k+1)$, with $(n)_0$ defined to be 1. Also, in a sequence of length k, a transition of the form $(\ldots,i,i+1,\ldots)$ is called a successor transition. We compute $C_{n,k}$ by using inclusion-exclusion on the number of successor transitions in sequences of length k.

Burnett and Coffman show that the number of sequences counted by $C_{n,k}$ with j initial successor transitions is equal to the number of sequences counted by $C_{n,k}$ with successor transitions in any j designated positions. Thus sequences of the form $(1,2,3,\ldots,j,j+1,\ldots)$ are equally numerous with sequences that have successor transitions in any of the $\binom{k-1}{j}$ ways that we can select j of the k-1 transitions. At this point our analysis departs from Burnett and Coffman.

Given that a sequence has j initial successor transitions, that is, a sequence of the form $(1,2,3,\ldots,j,j+1,\ldots)$, there are precisely

$$(n-j-1)(n-j-2)\ldots(n-k+1) = (n-j-1)_{k-j-1}$$

ways of selecting the remaining components so that the sequence contains no integer twice. Each of these sequences has at least j successor transitions, with the first j transitions guaranteed to be successor transitions. Since there are $\binom{k-1}{j}$ ways of selecting j out of k-1 transitions, we conclude that for each selection of j positions for successor transitions there are $(n-j-1)_{k-j-1}$ sequences with successor transitions in at least these j positions. For an inclusion-exclusion argument we define S_j to be:

$$S_j = \binom{k-1}{j} (n-j-1)_{k-j-1}$$

We let a_i denote the attribute of having a successor transition as the i^{th} transition, and we note that S_j enumerates all sequences with at least j attributes for every possible selection of the j attributes. Then inclusion-exclusion gives us the formula:

$$C_{n,k} = \sum_{j=0}^{k-1} (-1)^j S_j = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} (n-j-1)_{k-j-1} \qquad (1)$$

Some values of $C_{n,k}$ for small n and k are shown in Table I.

A discussion of a substantially similar problem appears in Liu [1968, pp. 110-111]. The numbers in Table I appear in Riordan [1958, p. 188] in a discussion of a problem related to the problem of derangements that is called the problem of rencontres.

When n=k, (1) takes the form:

$$C_{n,n} = \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} (n-j-1)! = (n-1)! \sum_{j=0}^{n-1} (-1)^j/j! \approx \frac{(n-1)!}{e}$$

This is the well-known solution for the number of derangements on n-1 letters. It is not difficult to compute the value of $C_{n,n-1}$ because of the formula

$$\begin{aligned} C_{n,n-1} &= C_{n,n} + C_{n-1,n-1} \\ &\approx \frac{(n-1)!}{e} + \frac{(n-2)!}{e} \\ &\approx \frac{n(n-2)!}{e} \\ &= \frac{1}{n-1} C_{n+1,n+1} \end{aligned}$$

As j increases, the magnitudes of the terms in (1) decrease so that we can bound (1) from above by its first term and from below by summing the first two terms. Thus we have

$$\begin{aligned} (n-1)_{k-1} \geq C_{n,k} &\geq (n-1)_{k-1} - (k-1)(n-2)_{k-2} \qquad (2) \\ &= (n-k)(n-2)_{k-2} \\ &= (n-2)_{k-1} \end{aligned}$$

Table I

	k = 1	2	3	4	5	6
n = 1	1	-	-	-	-	-
2	1	0	-	-	-	-
3	1	1	1	-	-	-
4	1	2	3	2	-	-
5	1	3	7	11	9	-
6	1	4	13	32	53	44

$C_{n,k}$

For k much less than n, the upper and lower bounds are rather close to each other, thus giving good estimates of $C_{n,k}$. As k approaches n, the upper bound approaches n-1 times the lower bound, until k=n, at which point the ratio become infinite. Consequently, with inequality (2), and the formulas for $C_{n,n}$ and $C_{n,n-1}$ we can estimate $C_{n,k}$ for all values of n and k to within a factor of n.

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