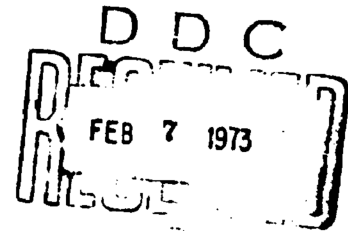


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**ASYMPTOTIC BOUNDS FOR THE NUMBER  
OF CONVEX  $n$ -OMINOES**

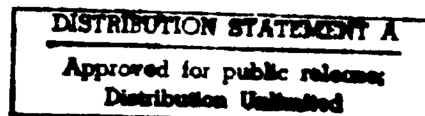
**BY**

**DAVID A. KLARNER  
AND  
RONALD L. RIVEST**

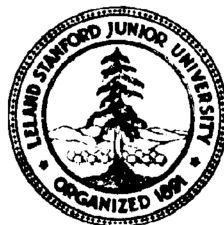


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## 13. ABSTRACT

Unit squares having their vertices at integer points in the Cartesian plane are called cells. A point set equal to a union of  $n$  distinct cells which is connected and has no finite cut set is called an n-omino. Two n-ominoes are considered the same if one is mapped onto the other by some translation of the plane. An n-omino is convex if all cells in a row or column form a connected strip. Letting  $c(n)$  denote the number of different convex n-ominoes, we show that the sequence  $((c(n))^{1/n}; n = 1, 2, \dots)$  tends to a limit  $\gamma$ , and  $\gamma = 2.309138\dots$ .

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Asymptotic Bounds for the Number  
of Convex  $n$ -Ominoes

by David A. Klarner and Ronald L. Rivest  
Computer Science Department, Stanford University

Introduction

Unit squares having their vertices at integer points in the Cartesian plane are called cells. A point set equal to a union of  $n$  distinct cells which is connected and has no finite cut set is called an  $n$ -omino. Two  $n$ -ominoes are considered the same if one is mapped onto the other by some translation of the plane. (Such  $n$ -ominoes were called fixed animals with  $n$  cells by R. C. Read [8].) For example, there are six different 3-ominoes as shown in Figure 1.

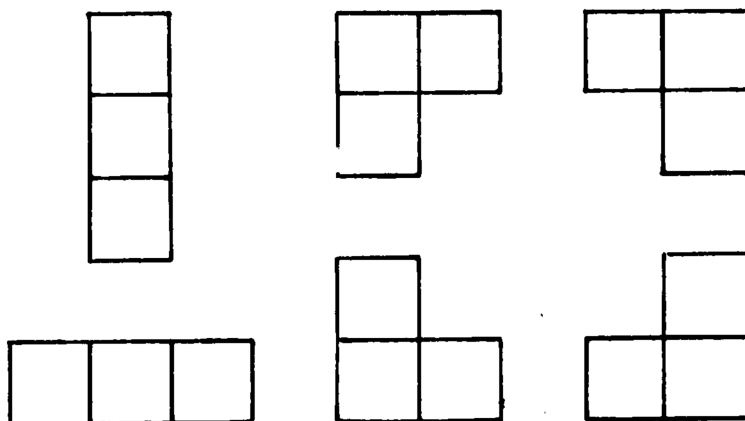


Figure 1. The 3-ominoes.

Let  $t(n)$  denote the number of distinct  $n$ -ominoes. It is known [2] that the sequence  $((t(n))^{1/n}; n = 1, 2, \dots)$  tends to a limit  $\theta$ . The investigation of  $\theta$  began with Eden's [1] work; he managed to prove that  $3.14 < \theta \leq 6.75$ . There has been considerable effort expended to improve these bounds. Currently, the best lower bound (given in [2]) is  $3.72 < \theta$ , while the best upper bound (given in [5]) is  $\theta < 4.65$ .

An  $n$ -omino is row-convex when each row of the  $n$ -omino is a connected strip of cells. Column-convex  $n$ -ominoes are defined analogously. All six of the 3-ominoes (shown in Figure 1) are both row-convex and column-convex; in general, such  $n$ -ominoes are said to be row-column-convex, or just convex for short. It was shown in [3] (and in [2] by a second method) that

$$(1) \quad \frac{x(1-x)^3}{1-4x+7x^2-5x^3} = \sum_{n=1}^{\infty} b(n)x^n$$

where  $b(n)$  denotes the number of distinct row-convex  $n$ -ominoes. (This result was also obtained by Polya [6].) Thus, it follows that the sequence  $((b(n))^{1/n}; n = 1, 2, \dots)$  tends to a limit  $\beta$  which is equal to the largest real root of  $y^3 - 4y^2 + 7y - 5 = 0$ ; that is,  $\beta = 3.20 \dots$

Recently, Donald Knuth wrote us from his sabbatical hide-out in CENSORED (where he is secretly writing Volume 4 of his septuple, The Art of Computer Programming), and asked us if the number  $c(n)$  of convex  $n$ -ominoes had been investigated. This paper is entirely motivated by Knuth's question. We shall be concerned with the problem of effectively calculating the limit  $\gamma$  of the sequence  $((c(n))^{1/n}; n = 1, 2, \dots)$ . One of the first things we prove is that this limit exists. Later on we show how to calculate upper and lower bounds for  $\gamma$  and give the best results obtained by these methods.

Existence of  $\lim_{n \rightarrow \infty} (c(n))^{1/n}$

Following Ceasar's admonition, we divide, then conquer. A convex  $n$ -omino may be split into three parts by making two cuts between certain rows so that the upper and lower parts are roughly trapezoids, and the middle part is roughly a parallelogram. A typical sectioning of this sort is shown in Figure 2. More precisely, the trisection of a convex  $n$ -omino  $A$  is accomplished by cutting along the lowest level of  $A$  where the left boundary of  $A$  goes to the right and by cutting along the lowest level of  $A$  where the right boundary of  $A$  goes to the left.

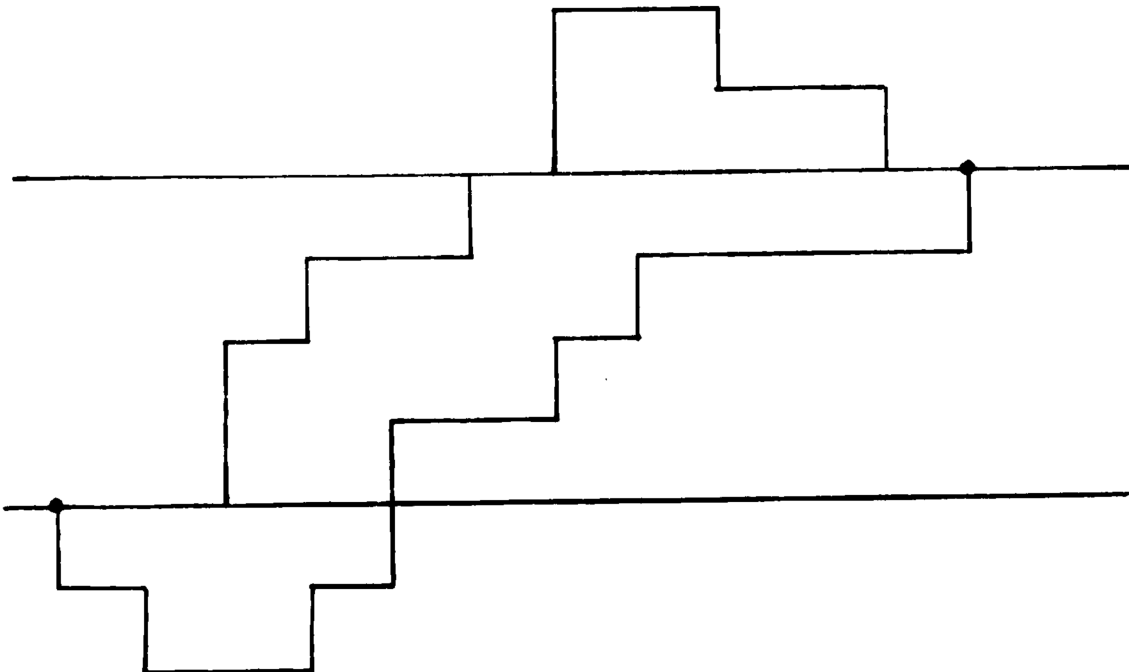


Figure 2. Trisection of a convex 28-omino.

A convex  $n$ -omino whose left boundary climbs to the right and whose right boundary climbs to the left corresponds to a partition of  $n$  called a stack by E. M. Wright [9]. We let  $s(n)$  denote the number of distinct  $n$ -ominoes corresponding to stacks; for example, there are four 3-ominoes shown in Figure 1 which correspond to stacks, so  $s(3) = 4$ . A convex  $n$ -omino whose left and right boundaries both climb to the right is called a parallelogram, and  $p(n)$  will denote the number of distinct  $n$ -ominoes which are parallelograms. Clearly,  $p(n) \leq c(n)$  for all  $n$ ; also,  $s(n) \leq p(n)$  for all  $n$  (the diagram in Figure 3 suggests a proof of this fact). Finally, an obvious construction establishes that  $p(m)p(n) \leq p(m+n)$  for all  $m, n$ . Now we use the fact that if  $\{u_n\}$  is a sequence of natural numbers such that  $((u_n)^{1/n}; n = 1, 2, \dots)$  is bounded and  $u_m u_n \leq u_{m+n}$  for all  $m, n$ , then  $\lim_{n \rightarrow \infty} (u_n)^{1/n}$  exists. (For similar results, see Pólya and Szego [7, p. 171].) We have  $p(n) \leq b(n) < (3.20)^n$  for all large  $n$ , and  $p(m)p(n) \leq p(m+n)$ , so

$$(2) \quad \lim_{n \rightarrow \infty} (p(n))^{1/n} = \gamma$$

exists. Using the fact that every convex  $n$ -omino splits into two stacks and one parallelogram, we can reconstruct these  $n$ -ominoes by pasting together two stacks and one parallelogram in various ways. Again, using an obvious construction, and using the fact that  $p(i)p(j)p(k) \leq p(i+j+k)$  for all  $i, j, k$ , it is easy to show that

$$\begin{aligned}
(3) \quad c(n) &\leq 2n^2 \sum_{(i,j,k)} s(i)p(j)s(k) \\
&\leq 2n^2 \sum_{(i,j,k)} p(i)p(j)p(k) \\
&\leq 2n^2 \binom{n+2}{2} p(n) \leq (n+2)^4 p(n)
\end{aligned}$$

where the index of summation in the sums extends over all compositions  $(i,j,k)$  of  $n$  into non-negative parts. There are  $\binom{n+2}{2}$  such compositions.

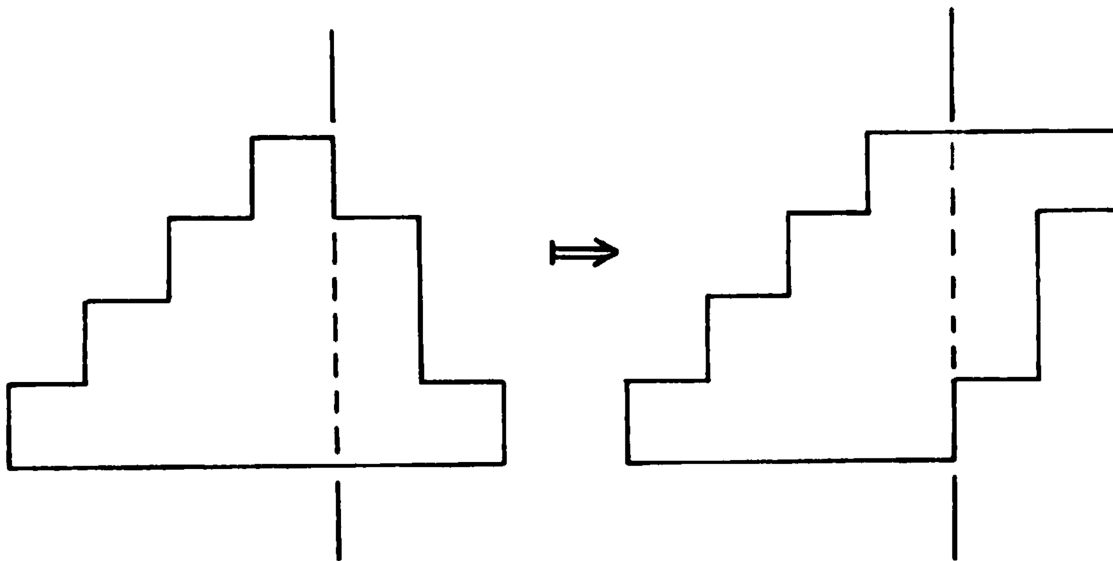


Figure 3. An injection showing  $s(n) \leq p(n)$ .

Using (2) and (3) together with the fact that  $p(n) \leq c(n)$  for all  $n$ , we have

$$\begin{aligned}
(4) \quad \gamma &= \lim_{n \rightarrow \infty} (p(n))^{1/n} \leq \liminf_{n \rightarrow \infty} (c(n))^{1/n} \\
&\leq \limsup_{n \rightarrow \infty} (c(n))^{1/n} \leq \lim_{n \rightarrow \infty} ((n+2)^4 p(n))^{1/n} = \gamma.
\end{aligned}$$



Hence,  $\lim_{n \rightarrow \infty} (c(n))^{1/n}$  exists, and

$$(5) \quad \lim_{n \rightarrow \infty} (c(n))^{1/n} = \lim_{n \rightarrow \infty} (p(n))^{1/n} = \gamma.$$

### An Integral Equation

We shall use a theory developed in [4] concerning a double sequence  $(b(n,a): n, a = 1, 2, \dots)$  defined in terms of given sequences  $(f(m,n): m, n = 1, 2, \dots)$  and  $(g(n): n = 1, 2, \dots)$  as follows:

$$(6) \quad b(n,a) = \sum f(a_1, a_2) f(a_2, a_3) \dots f(a_{k-1}, a_k) g(a_k)$$

where the index of summation extends over all  $k$ -tuples  $(a_1, \dots, a_k)$  of natural numbers for  $k = 1, \dots, n$  with  $a_1 = a$  and  $a_1 + \dots + a_k = n$ .

It was shown that if

$$(7) \quad G(x) = \sum_{n=1}^{\infty} g(n) x^n,$$

and

$$(8) \quad F(x,y) = \sum_{m,n=1}^{\infty} f(m,n) x^m y^n,$$

converge for  $|x|$  and  $|y|$  sufficiently small, then

$$(9) \quad B(x,y) = \sum_{n=1}^{\infty} \sum_{a=1}^n b(n,a) y^a x^n$$

converges for  $|x|$  and  $|y|$  sufficiently small, and

$$(10) \quad B(x,y) = G(xy) + \frac{1}{2\pi i} \int_C F(xy, \frac{1}{s}) B(x,s) \frac{ds}{s}$$

where  $C$  is a contour in the  $s$ -plane which includes  $s = 0$  and the singularities of  $F(xy, \frac{1}{s})$  but excludes the singularities of  $B(x, s)$ . The theory of (10) runs parallel to that of the Fredholm integral equation. In particular, if  $F(x, y)$  has the special form

$$(11) \quad F(x, y) = R_1(x)S_1(y) + \dots + R_t(x)S_t(y) ,$$

we say  $F$  is separable, and it turns out that (10) can be converted into a system of  $t$  equations linear in  $t$  unknown functions. The system can be solved and the solution yields a formula for  $B(x, y)$ . We shall give an example of this later on.

If  $F$  is not separable we can still get information about  $B$  by approximating  $F$  with something that is separable. Suppose

$$(12) \quad K(x, y) = \sum k(m, n)x^m y^n$$

and  $k(m, n) \leq f(m, n)$  for all  $m, n$ , then we say  $K$  is a lower bound on  $F$ ; an upper bound on  $F$  is defined analogously. If  $K$  is separable, we may substitute  $K$  for  $F$  in (10) and calculate a lower bound for  $B$ . Upper bounds for  $B$  may be obtained in a similar fashion. We shall adopt this strategy too, so an example is forthcoming.

The relevance of the foregoing discussion to the enumeration of  $n$ -celled parallelograms is as follows: the number of  $(m+n)$ -celled parallelograms having  $m$  cells in one row and  $n$  cells in a second row is

$$(13) \quad f(m, n) = \min\{m, n\} .$$

It is fairly easy to show that the number of  $n$ -celled parallelograms with exactly  $k$  rows of cells having exactly  $a_i$  cells in the  $i$ -th row for  $i = 1, \dots, k$  is

$$(14) \quad f(a_1, a_2) f(a_2, a_3) \dots f(a_{k-1}, a_k) \quad .$$

Thus, if we take  $f$  as defined in (13) and put  $g(j) = 1$  for all  $j$ , we can sum (6) over  $a = 1, \dots, n$  and obtain  $p(n)$ . In this case, we have

$$(15) \quad F(x, y) = \frac{xy}{(1-x)(1-y)(1-xy)} \quad ,$$

and

$$(16) \quad G(x) = \frac{x}{1-x} \quad .$$

Substituting these functions in (10) gives

$$(17) \quad B(x, y) = \frac{xy}{1-xy} + \frac{1}{2\pi i} \int_C \frac{xy B(x, s) ds}{(1-xy)(s-1)(s-xy)}$$

$$= \frac{xy}{1-xy} + \frac{xy}{(1-xy)^2} B(x, 1) - \frac{xy}{(1-xy)^2} B(x, xy) \quad .$$

We can iterate (17) to eliminate  $B(x, xy), B(x, x^2y), \dots$  successively to find

$$(18) \quad B(x, y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{k(k+1)/2} y^k (1 - x^k y + B(x, 1))}{(1-xy)^2 (1-x^2y)^2 \dots (1-x^k y)^2} \quad .$$

Setting  $y = 1$  in (18), we solve for  $B(x, 1)$ , the generating function of  $(p(n): n = 1, 2, \dots)$ , which turns out to be

$$(19) \quad B(x, 1) = \frac{\frac{x}{1-x} - \frac{x^3}{(1-x)^2(1-x^2)} + \frac{x^6}{(1-x)^2(1-x^2)^2(1-x^3)} - \dots}{1 - \frac{x}{(1-x)^2} + \frac{x^3}{(1-x)^2(1-x^2)^2} - \frac{x^6}{(1-x)^2(1-x^2)^2(1-x^3)^2} + \dots}$$

$$= \sum_{n=1}^{\infty} p(n) x^n \quad .$$

We have been unable to make use of (19) in estimating  $p(n)$  . Instead we use upper and lower bounds for  $F$  as defined in (15), and then use (10) to calculate upper and lower bounds for  $B$  .

### Lower Bounds

Let

$$(20) \quad F_k(x,y) = \sum_{m,n=1}^k f(m,n)x^m y^n$$

where  $f(m,n) = \min\{m,n\}$  just as in (13), and let  $B_k(x,y)$  denote the solution of (10) having  $F_k$  substituted for  $F$  . Since  $F_k$  is a lower bound for  $F$  , it follows that  $B_k$  is a lower bound for  $B$  . It was shown in [4] that when the kernel of (10) is approximated by a polynomial as in this case, then  $B_k(x,1)$  is a rational function, say  $B_k = P_k/Q_k$  with  $P_k$  and  $Q_k$  polynomials, and the denominator of  $B_k$  may be expressed as a determinant. In the present situation this turns out to be

$$(21) \quad Q_k(x) = \begin{vmatrix} 1-x & 1 & 1 & \dots & 1 \\ 1 & 2-x^2 & 2 & \dots & 2 \\ 1 & 2 & 3-x^3 & \dots & 3 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 2 & 3 & \dots & k-x^k \end{vmatrix}$$

If we put  $Q_0(x) = 1$  and  $Q_1(x) = 1-x$  we can use (21) to verify that

$$(22) \quad Q_k(x) = (1-x^{k-1}-x^k)Q_{k-1}(x) - x^{2k-2}Q_{k-2}(x)$$

for  $k = 2, 3, \dots$  . For example,

$$Q_2(x) = 1 - 2x - x^2 + x^3 ,$$

$$Q_3(x) = 1 - 2x - 2x^2 + 2x^3 + 2x^4 + x^5 - x^6 ,$$

$$Q_4(x) = 1 - 2x - 2x^2 + x^3 + 3x^4 + 5x^5 - 2x^6 - 2x^7 - 2x^8 - x^9 + x^{10} .$$

Letting  $\gamma_k$  denote the largest real root of  $Q_k(1/x) = 0$  , we have  $\gamma_1 \leq \gamma_2 \leq \dots \leq \gamma$  , where  $\gamma$  is defined in (2). We have used a computer to calculate lower bounds for  $\gamma_1, \gamma_2, \dots, \gamma_{10}$  given in the table. Our results indicate that the sequence  $\{\gamma_k\}$  converges very quickly to the value 2.30913859... , our best lower bound for  $\gamma$  .

#### Upper Bounds

For  $k = 1, 2, \dots$  we define upper bounds  $r^k(m, n)$  for  $f(m, n) = \min\{m, n\}$  as follows:

$$(23) \quad r^k(m, n) = \begin{cases} m , & \text{if } k < n < m , \\ f(m, n) , & \text{otherwise.} \end{cases}$$

Hence,

$$(24) \quad F^k(m, n) = \sum_{m, n=1}^{\infty} r^k(m, n) x^m y^n \\ = \frac{xy}{(1-x)^2(1-y)} - \frac{x^2y}{(1-x)^2} - \dots - \frac{x^{k+1}y^k}{(1-x)^2}$$

is an upper bound for  $F$  ; furthermore, note that  $F^k$  is separable.

Let  $B^k$  denote the solution of (10) with  $F^k$  substituted for  $F$  .

Then,

$$(25) \quad B^k(x, y) = \frac{xy}{1-xy} + \frac{xyB^k(x, 1)}{(1-xy)^2} - \frac{xy}{(1-xy)^2} \sum_{r=1}^k x^r y^r B_r^k(x)$$

where

$$B_r^k(x) = \frac{1}{k!} \frac{\partial^r}{\partial s^r} B^k(x, s) \Big|_{s=0}.$$

Now we use (25) to get a system of equations involving  $B_1^k, \dots, B_k^k$ . Take the  $r$ -th partial derivative with respect to  $y$  at  $y = 0$  and divide by  $r!$  in (25) to get

$$(26) \quad B_r^k(x) = x^r + rx^r B^k(x, 1) - \sum_{j=1}^{r-1} (r-j)x^r B_j^k(x),$$

from which it follows that

$$(27) \quad B_{r+1}^k(x) = (2x - x^{r+1})B_r^k(x) - x^2 B_{r-1}^k(x).$$

Setting  $B_r^k(x) = P_r(x) + Q_r(x)B^k(x, 1)$  for  $r = 1, \dots, k$ , it follows that  $P_r$  and  $Q_r$  also satisfy the difference equation (27). Also, we can substitute  $P_r + Q_r B^k$ . For  $B_r$  in (25) with  $y = 1$  and solve for  $B^k(x, 1)$  in terms of  $P_1, Q_1, \dots, P_k, Q_k$  to obtain

$$(28) \quad B^k(x, 1) = \frac{x - x^2 - \sum_{j=1}^k x^{j+1} P_j(x)}{1 - 3x + x^2 + \sum_{j=1}^k x^{j+1} Q_j(x)}.$$

Thus,  $B^k$  is a rational function whose numerator  $N_k$  and denominator  $D_k$  we know how to compute because they are defined in terms of  $P_1, \dots, P_k$  and  $Q_1, \dots, Q_k$  which we know how to compute. Let  $\beta_k$  denote the largest real root of  $D_k(1/x)$ , then we know

$$(29) \quad \lim_{n \rightarrow \infty} \left( \sum_{a=1}^n b^k(n, z) \right)^{1/n} = \beta_k \leq \gamma,$$

and  $\beta_1 \geq \beta_2 \geq \dots \geq \gamma$ . Thus, we can calculate upper bounds for  $\beta_1, \beta_2, \dots$  to obtain successively better upper bounds for  $\gamma$ .

Using the definitions

$$(30) \quad D_k = 1 - 3x + x^2 + x^2 Q_1 + \dots + x^{k+1} Q_k ,$$

$$(31) \quad Q_{r+1} = (2x - x^{r+1})Q_r - x^2 Q_{r-1} \quad (r > 1) ,$$

and  $Q_1 = x$ ,  $Q_2 = 2x^2 - x^3$ , the polynomials  $D_1, D_2, \dots$  are calculated with relative ease. For example, we found

$$\begin{aligned} D_1 &= 1 - 3x + x^2 + x^3 , \\ D_2 &= 1 - 3x + x^2 + x^3 + 2x^5 - x^6 , \\ D_3 &= 1 - 3x + x^2 + x^3 + 2x^5 - x^6 + 3x^7 - 2x^8 - 2x^9 + x^{10} . \end{aligned}$$

Using a computer, the polynomials  $D_1, \dots, D_{10}$  were calculated via (30), and upper bounds for  $\beta_k$ , the largest real root of  $D_k(1/x) = 0$ , were computed for  $1 \leq k \leq 10$  using the Newton-Raphson method. These upper bounds for  $\beta_k$  are given in the table.

Combining our upper and lower bounds we can conclude that

$$(32) \quad \gamma = \lim_{n \rightarrow \infty} (c(n))^{1/n} = 2.309138\dots .$$

$k$	$\gamma_k$	$\beta_k$
1	1.00000000	2.41421356
2	2.24697960	2.33578290
3	2.30855218	2.31475605
4	2.30913772	2.31023504
5	2.30913859	2.30934711
6	2.30913859	2.30917790
7	2.30913859	2.30914598
8	2.30913859	2.30913998
9	2.30913859	2.30913885
10	2.30913859	2.30913864

**Table**



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