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TWO PAPERS ON PARTIAL PREDICATE CALCULUS

BY

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ABSTRACT: These papers, published in 1938 and 1943, contain the first treatment of a logic of partial predicates. Bochvar's treatment is of current interest for two reasons. First, partial predicate and function logic are important for mathematical theory of computation because functions defined by programs or by recursion cannot be guaranteed to be total. Second, natural language may be better approximated by a logic in which some sentences may be undefined than by a conventional logic. Bochvar's use of his system to avoid Russell's paradox is of interest here, and in partial predicate logic it may be possible to get more of an axiomatization of truth and knowledge than in conventional logic.

The papers translated are "On a three-valued logical calculus and its application to the analysis of contradictions," Recueil Mathematique, N.S.4 (1938), pp. 287-308, and "On the consistency of a three-valued logical calculus," ibid, 12 (1943), pp. 353-369.

We also print a review and a correction by Alonzo Church that appeared in the Journal of Symbolic Logic. The review was in vol. 4, 2 (June 1939), p. 99, and the additional comment was in vol. 5, 3 (September 1940), p. 119.

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A THREE-VALUED LOGICAL CALCULUS AND ITS APPLICATION TO THE ANALYSIS OF THE PARADOXES OF THE CLASSICAL EXTENDED FUNCTIONAL CALCULUS

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[From Matyematichyeski Sbornik (Recueil Mathematique), N.S. 4 (1938), pp. 287-308.]

The three-valued system to which this study is devoted is of interest as a logical calculus for two reasons: first, it is based on formalization of certain basic and intuitively obvious relations satisfied by the predicates "true", "false" and "meaningless" as applied to propositions, and as a result the system possesses a clear-cut and intrinsically logical interpretation; second, the system provides a solution to a specifically logical problem, analysis of the paradoxes of classical mathematical logic, by formally proving that certain propositions are meaningless.

The paper consists of three parts. In the first we develop the elementary part of the system -- the propositional calculus -- on the basis of intuitive considerations. In the second part we outline the "restricted" functional calculus corresponding to the propositional calculus. The third and last part uses a certain "extension" of the functional calculus to analyze the paradoxes of classical mathematical logic.

We are indebted to Professor V.I. Glivenko for much valuable advice and criticism. In particular, he provided a more suitable definition of the function a (see I, Section 2, subsection 1.).

[Typist's note: Subscripts are indicated by ↓, and • is used in place of the logical connective ↓ used in the original. S is used for capital Sigma.]

I

PROPOSITIONAL CALCULUS

In order to clarify the basic features of the propositional calculus, let us analyze the intuitive properties of the basic types of propositions.

First, however, we shall rigorously determine the relation between "proposition" and "sentence". Following accepted usage, we shall say that a proposition is meaningful if it is true or false. Morever, a proposition will be called a sentence if and only if it is meaningful; otherwise we shall call the proposition meaningless. Any sentence is clearly a proposition. Any proposition is either meaningless, true, or false. If a proposition A is meaningless then the propositions "A is false" and "A is true" are meaningful but false. The predicates "true," "false" and "meaningless" may be applied to any proposition.

Now let A and B be any propositions. Consider the following propositions:

I	II
"A"	"A is valid"
"not-A"	"A is false"
"A and B"	"A is valid and B is valid"
"A or B"	"A is valid or B is valid"
"If A, then B"	"If A is valid, then B is valid"

We shall call types I and II internal and external forms of affirmation, negation, conjunction, disjunction and implication, respectively. "A is meaningless" is obviously an external form which does not correspond to any internal form.

It is clear that any internal form and its corresponding external form have different "meanings." The essential difference between internal and external forms is easily indicated by letting A (or B) be a meaningless proposition. First consider internal forms. It seems quite obvious that if A is a meaningless proposition then "not-A" is also meaningless; similarly, it is intuitively clear that any combination of a meaningless proposition A and a proposition B by the operations "--- and ---", "--- or ---", "if ---, then ---" can only yield a new meaningless proposition.

The situation is quite different for external forms. Let A be a meaningless proposition. Then, obviously, its external affirmation "A is valid" is false, but not meaningless. Similarly, the external negation "A is false" is false, but not meaningless, if A is meaningless. It is easily seen that the other external forms are also never meaningless when A is a meaningless proposition.

In fact, the external forms (conjunction, disjunction and implication) are precisely the corresponding internal forms with A and B replaced by their external affirmations. Now, since an external affirmation is never meaningless, it is obvious that this must be true of the external conjunction, external disjunction and external implication. (1)

Clearly, the external forms of sentences are formally equivalent to the corresponding internal forms. In other words, the internal and external forms of a sentence are either both true or both false.

This is a partial explanation of the ambiguous intuitive interpretation, still widespread in the literature of mathematical logic, of the primitive connectives of the classical sentential calculus (2), viz., internal and external forms are employed interchangeably for negation, conjunction, disjunction and implication (see, for example, PRINCIPIA MATHEMATICA, Vol. I, Part 1, Section A). However this ambiguity has nothing to do with the actual nature of the classical formal sentential calculus. Indeed, the classical sentential calculus does not regard affirmations as functions of a sentential variable, i.e., it considers only internal affirmations and therefore admits interpretation only via a system of internal forms.

We must admit that, in principle, the system of internal forms is of course absolutely adequate for an intuitive interpretation of the formalism of classical logic and mathematics, since the latter deals with the symbols of the sentential calculus. Owing to the incompleteness of natural language it is rather difficult to find a brief and convenient verbal expression for the internal negation of a sentence of the type "A and B"; nevertheless, in principle it is quite clear that this internal negation indeed exists and is even easily expressed in terms of natural language, provided one resorts to certain definitions, which in themselves are quite legitimate.

Accordingly, internal and external forms will be referred to as classical and nonclassical intuitive functions of propositional variables, respectively.

SECTION 2. Truth-table form of propositional calculus

1. BASIC CONCEPTS AND DEFINITIONS. a, b, c, d, ... will be propositional variables. The set of values for each of these variables comprises three elements: T (read "true"), F (read "false") and U (read "meaningless") and no others.

We introduce the usual functions of the propositional variables. Each function is defined by a truth table, as follows. First list all possible systems of values for the arguments, in an arbitrary but fixed order, to the left of the double line; to the right of the double line enter the values of the function.

As the primitive classical functions we introduce formal internal negation, ¬a (read "not-a") and formal internal conjunction a∩b (read "a and b"), defined by the following truth tables:

a	¬a
T	F
F	T
U	U

a	b	a∩b
T	T	T
T	F	F
F	T	F
F	F	F
T	U	U
U	T	U
F	U	U
U	F	U
U	U	U

Our primitive nonclassical functions will be formal external affirmation ⊢ a (read "a is valid") and formal external negation >a (read "a is false"), defined by the following truth tables:

a	⊢a
T	T
F	F
U	F

a	>a
T	F
F	T
U	F

The following definitions are intended solely to simplify the notation and need no explanation (the symbol $=_D$ denotes equality by definition):

$$¬¬a =_D ¬(¬a),$$

$$¬ ⊢a =_D ¬(⊢a),$$

$$⊢ >a =_D ⊢ (>a),$$

$$>>a =_D >(>a)$$

and so on, for any finite sequence of symbols ¬, ⊢, > and the symbol

† to be introduced below.

We now define some classical functions using classical negation and conjunction:

$$(D{\cdot}1)\quad (a \cup b) =_D \neg(\neg a \cap \neg b)$$

$$(D{\cdot}2)\quad (a \supset b) =_D \neg(a \cap \neg b)$$

$$(D{\cdot}3)\quad (a \supset\subset b) =_D [(a \supset b) \cap (b \supset a)]$$

The function $a \cup b$ --- formal internal or classical disjunction --- is read "a or b". The function $a \supset b$ --- formal internal or classical implication --- is read "if a, then b". The matrices of the functions $a \cup b$, $a \supset b$, $a \supset\subset b$ are easily constructed using the above definitions.

Using formal external affirmation and formal external negation, we define the following functions:

$$(D{\cdot}4)\quad (a \wedge b) =_D (\vdash a \cap \vdash b)$$

$$(D{\cdot}5)\quad (a \vee b) =_D (\vdash a \cup \vdash b)$$

$$(D{\cdot}6)\quad (a \rightarrow b) =_D (\vdash a \supset \vdash b)$$

$$(D{\cdot}7)\quad (a \leftrightarrow b) =_D [(a \rightarrow b) \cap (b \rightarrow a)]$$

$$(D{\cdot}8)\quad (a \equiv b) =_D [(a \leftrightarrow b) \cap (\neg a \leftrightarrow \neg b)]$$

$$(D{\cdot}9)\quad \dagger a =_D \neg(\vdash a \cup > a)$$

$$(D{\cdot}10)\quad \bar{a} =_D - \vdash a \qquad (3)$$

The function $a \wedge b$ --- formal external or nonclassical conjunction --- is read "a is valid and b is valid". The function $a \vee b$ --- formal external or nonclassical disjunction --- is read "a is valid or b is valid". The function $a \rightarrow b$ --- formal external or nonclassical implication --- is read "if a is valid, then b is valid", or "the proposition b follows from the proposition a". The function $a \leftrightarrow b$ is read "a is equipotent to b". The function $a \equiv b$ is read "a is equivalent to b".

It is interesting to compare equipotence and equivalence. If $a \leftrightarrow b$, then the truth of either of the propositions a,b implies the truth of the other, but this does not mean that a and b are logically equivalent. If one of them is false, the other need not be false --- it may be meaningless. One cannot infer from

$$a \leftrightarrow b$$

that

$$\neg a \leftrightarrow \neg b$$

or that

$$\dagger a \leftrightarrow \dagger b.$$

On the other hand, any proposition following from a also follows from b, and vice versa, and in this sense a and b are equipotent.

If

$$a \equiv b,$$

not only does the truth of either of a,b imply that of the other; now, in addition, if one is false, so is the other, and if one is meaningless, so is the other. If two propositions are equivalent, they must be equipotent, but the converse is generally false. Note that the truth tables of equivalent functions are identical. Hence equivalence plays the part of "mathematical identity" in the propositional calculus. The function †a is read "a is meaningless".

Finally, the function $\bar{a}$ is read "a is not valid".

Using the definitions, we construct the truth tables of †a and $\bar{a}$:

a	†a
T	F
F	F
U	T

a	$\bar{a}$
T	F
F	T
U	T

The truth tables of the functions a∧b, a∨b, a→b, a↔b, a≡b are also easily constructed.

We now give a rigorous definition of formula. The definition is inductive:

1) Any propositional symbol is a formula.
2) If A is a formula, then ¬A, |¬A and >A are formulas.
3) If A and B are formulas, then A∩B is a formula.

To simplify the notation for formulas, we shall use the "dot" notation [of Principia Mathematica].

The symbols ⊃, ⊃⊂, →, ↔, ≡, $\equiv_D$ are stipulated to be of equal rank, higher than that of the symbols ∩, ∪, ∧, ∨. The latter bind more strongly than the former.

The symbols ¬, |¬, >, † act only on the letters and parentheses directly following them.

The symbol ¯ always applies to the entire expression below it.

Thus,

$$a \vee b \rightarrow b \vee a$$

denotes the formula

$$(a \vee b) \rightarrow (b \vee a).$$

The formula

a → b .∩. a ↔ ¬b ∨ †b .→ ¬a ∨ †a

denotes the formula

$((a \rightarrow b) \cap [a \rightarrow (\neg b \vee \uparrow b)]) \rightarrow (\neg a \vee \uparrow a)$,

The definition of the function $a \leftrightarrow b$ may now be written

$$a \leftrightarrow b \;.\underset{D}{=}.\; a \rightarrow b \;.\cap.\; b \rightarrow a,$$

and so on.

A formula is said to be provable in the truth table propositional logic if it has the value T for all possible argument values. Provable formulas are also known as tautologies. Proof is reduced to verification. Verification is most systematically and simply carried out by constructing a truth table for the function in question.

A formula which does not take the value T for any values of its arguments is known as a contradiction. If A is a contradiction, then $\bar{A}$ is a tautology. Moreover, if one of the formulas $\neg A$, $\supset A$, $\uparrow A$, $\bar{\bar{A}}$, is provable, then A is a contradiction.

A formula which contains only propositional variables and symbols for the classical functions will be called a classical formula. Let Phi(a↓1,...,a↓n) be any formula, with a given truth table. The truth table has 3↑n rows. Call the set of rows in which no argument ever assumes the value U the TF-subtable of the formula. It is clear that the TF-subtable contains 2↑n rows. The remaining rows comprise what we shall call the U-subtable of the function.

The U-subtable contains 3↑n - 2↑n rows.

2. UNPROVABLE FORMULAS IN THE PROPOSITIONAL CALCULUS.

THEOREM I. No classical formula is provable in the propositional calculus.

PROOF. Obvious, since any every classical formula assumes the value U when one of its arguments assumes the value U.

THEOREM II. No contradiction is provable in the propositional calculus.

PROOF. This follows immediately from the definitions of provable formula and contradiction in subsection 1.

Examples of contradictions are the formulas

$$a \wedge > a,$$
$$a \equiv > a,$$
$$a \leftrightarrow \vec{a},$$

THEOREM III. No formula whose construction involves only nonclassical functions can be equivalent to a classical formula.

THEOREM IV. The formula ↑a (therefore also $\bar{A}$) cannot even be equipotent to a classical formula.

Theorems III and IV follow immediately from the form of the truth tables 1 - 6 in subsection 1.

3. IMPORTANT FORMULAS PROVABLE IN THE PROPOSITIONAL CALCULUS.

THEOREM V. Any formula provable in the classical sentential calculus which has the form A ⊃ B (4), where A and B contain the same variables, remains provable in the nonclassical calculus if the symbol ⊃ between A and B is replaced by → and the variables are regarded as propositional variables.

Similarly, any formula provable in the classical sentential calculus which has the form A ⊃⊂ B, where A and B contain the same variables, remains provable in the propositional calculus if the symbol ⊃⊂ between A and B is replaced by ≡ and the variables regarded as propositional variables.

We prove the first part of the theorem. It is obvious that if A ⊃ B is provable in the classical sentential calculus then every row in the TF-subtable of the formula A→B assigns the function the value T.

Now let one of the variables a·i assume the value U. Since A and B are classical formulas and both contain a·i by assumption, they both assume the value U. But by the definition of a→b,

U→U = T.

Consequently, every row in the U-subtable of the formula A→B also gives the formula the value T. This proves the theorem.

THEOREM VI. The truth table propositional calculus contains a subsystem isomorphic to the classical truth table sentential calculus; the formulas of this subsystem are derived from those of the classical sentential calculus by the following transformations (5) (we abbreviate "classical sentential calculus" by c.s.c., and "propositional calculus" by p.c.):

1) Replace each sentential variable by the propositional variable with the same symbol;
2) replace the c.s.c. symbol ‾ by the p.c. symbol ‾ ;
3) " " " " ∩ " " " " ∩;
4) " " " " ∪ " " " " ∨;
5) " " " " ⊃ " " " " →;
6) " " " " ⊃⊂ " " " " ".

PROOF. It is easily shown by truth tables that the following formulas are tautologies:

(1) a → a ∩ a
(2) a ∩ b → b ∩ a
(3) a → b .→. a ∩ c → b ∩ c
(4) a → b .∩. b → c .→. a → c
(5) b → .a → b
(6) a ∩ .a → b. → b
(7) a → a ∨ b
(8) a ∨ b → b ∨ a

(9) $a \to c \;,\cap,\; b \to c \;,\to,\; a \vee b \to c$

(10) $\bar{a} \to \;, a \to b$

(11) $a \to b \;,\cap,\; a \to \bar{b} \;,\to\; \bar{a}$

(12) $a \vee \bar{a}.$

The system of formulas (1) to (12) is an isomorphic image of the following system of c.s.c. formulas:

$a \supset a \cap a$

$a \cap b \supset b \cap a$

$a \supset b \;,\supset,\; a \cap c \supset b \cap c$

$a \supset b \;,\cap,\; b \supset c \;,\supset,\; a \supset c$

$b \supset,\; a \supset b$

$a \cap,\; a \supset b \;,\supset b$

$a \supset a \cup b$

$a \cup b \supset b \cup a$

$a \supset c \;,\cap,\; b \supset c \;,\supset,\; a \cup b \supset c$

$\neg a \supset,\; a \supset b$

$a \supset b \;,\cap,\; a \supset \neg b \;,\supset \neg a$

$a \cup \neg a.$

But, as is well known (6), this is an axiom system for the classical sentential calculus, if the rules of inference are as follows:

1) Principle of deduction [modus ponens]:

If a and a⊃b are provable formulas, then b is a provable formula.

2) Rule of combination:

If a and b are provable formulas, then a∩b is a provable formula.

3) The substitution principle in its conventional form.

We now deduce from the truth table of the function a→b that the deduction principle is valid in the propositional calculus in the following form:

If a and a→b are provable formulas, then b is a provable formula.

Furthermore, examination of the truth table of the function a∩b shows that the combination rule also holds in the propositional calculus:

If a and b are provable formulas, then a∩b is a provable formula.

Finally, it is obvious that the substitution principle also remains valid in the propositional calculus. This completes the proof of Theorem VI.

The isomorphic image of the classical sentential calculus whose existence we have just established will be called K+1. It would be easy to show that the propositional calculus contains another system isomorphic to sentential logic; it is obtained from K+1 by replacing the symbol ∩ by ∧ and the symbol → by ≡. This second isomorphic image of the classical sentential calculus will be called the system K+2 (7).

Theorems V and VI provide a tool for defining various classes of formulas provable in the propositional calculus. Thus, Theorem V is illustrated by the formulas:

(13) $a \equiv \neg\neg a$

(14) $\neg(a \cap b) \equiv \neg a \cup \neg b$

(15) $\neg(a \cup b) \equiv \neg a \cap \neg b$

(16) $\neg(a \supset b) \equiv a \cap \neg b$

(17) $a \supset \neg a \ . \rightarrow \neg a.$

However, one should not overrate the operative force of these formulas; for the formulas on either side of the symbols $\equiv$, $\rightarrow$ are classical formulas, and hence Theorem I (subsection 2) is relevant.

We now proceed to consider several additional, very important formulas of the propositional calculus. We first indicate the basic formulas which give the relation between classical and nonclassical formulas:

(18) $a \leftrightarrow \vdash a$

(19) $\neg a \leftrightarrow\ > a$

(20) $a \cap b \leftrightarrow b \cap a$

(21) $a \cup b \leftrightarrow b \cup a$

(22) $a \supset b \;.{\rightarrow}.\; a \rightarrow b.$

It is extremely important to note that the last two formulas involve only nonclassical implication (in one direction), while the others involve equipotence.

The next two formulas give the relation between meaninglessness on the one hand and classical and nonclassical negations on the other:

(23) $\uparrow a \equiv\ \uparrow \neg a$

(24) $\uparrow a \equiv\ >> a.$

Formula (25) shows that the external affirmation of a meaningless proposition is false:

(25) $\uparrow a \rightarrow\ > \vdash a.$

The following formulas are interesting:

(26) $\overline{\neg(a \cup \neg a)}$

(27) $>>(a \cup \neg a)$

(28) $>\uparrow(a \vee > a)$

(29) $\uparrow(a \cup \neg a) \equiv\ >(a \vee > a)$

(30) $\uparrow(a \cup \neg a) \equiv\ \uparrow a.$

One sees from formula (26) that the classical negation of the classical form of "tertium non datur" is always false or meaningless. Formula (27) states that the nonclassical negation of the classical "tertium non datur" is always false.

Formula (28) states that the nonclassical "tertium non datur" cannot be meaningless, i.e., the proposition stating that it is meaningless is always false.

Formula (29) expresses the fact that the classical "tertium non datur" is meaningless if and only if the nonclassical form is false. Finally, formula (30) states that the classical "tertium non datur" is meaningless if and only if the proposition itself is meaningless.

The following formulas will be particularly important for the analysis of paradoxes:

(31) $a \equiv \neg a \;.{\equiv}\; \uparrow a$

(32) $a \leftrightarrow \neg a \;.{\equiv}\; \uparrow a$

(33) $\neg \uparrow a \rightarrow.\; a \leftrightarrow \neg a \;:{\equiv}\; \uparrow a$

(34) $a \cup \neg a \rightarrow.\; a \leftrightarrow \neg a \;:{\equiv}\; \uparrow a$

Note also the following formulas:

(35) $a \leftrightarrow\ >a \ .\equiv\ \dagger a$

(36) $\vdash a \equiv\ >a \ .\equiv\ \dagger a$

However, the formula

$$a \equiv\ >a \ .\equiv\ \dagger a$$

is not valid. If a is meaningless, then $\dagger a$ is valid, but $a \equiv\ >a$ is always false.

By analogy, note the formulas

(37) $a \equiv \dagger a \ .\equiv\ >a$

(38) $a \equiv \dagger a \ .\leftrightarrow\ \neg a$

Also important are the formulas

(39) $\neg a \rightarrow .\ a \rightarrow b$

(40) $>a \rightarrow .\ a \rightarrow b$

(41) $\dagger a \rightarrow .\ a \rightarrow b$

Finally, we present the formulas

(42) $a \equiv b \ .\rightarrow.\ \neg a \equiv \neg b$

(43) $a \equiv b \ .\rightarrow.\ \dagger a \equiv \dagger b$

II

RESTRICTED FUNCTIONAL CALCULUS

SECTION 1. Basic concepts, notation and definitions

The variables of the functional calculus fall into three groups:

1) propositional variables: a, b, c, ...
2) object variables: x, y, z, ...
3) variables for functions of any finite number of object variables: f( ), g( ),....,phi( ), psi( ),....

Corresponding to these three groups of variables there are three groups of constants; notation for these will be introduced as the need arises.

The symbol f(x) is read: "x has property f". The symbol f(x,y) is read: "x stands in relation f to y". The symbol (x), the basic quantifier, is called the universal symbol. The symbol (x)f(x) is read: "all x have property f".

The concept of "formula" (sometimes also called an "expression") is defined inductively by the following rules:

1) Every propositional symbol is a formula.

2) Any function symbol in which the argument places are occupied by names of objects or symbols of object variables is a formula.

3) If A is a formula and A contains x as a free variable (depends on x) then (x)A is a formula.

4) If A is a formula, then ¬A, >A, |-A are formulas.

5) If A and B are formulas, then A∩B is a formula.

6) If a subformula is in the scope of a universal symbol for a certain variable, it cannot be in the scope of any other universal symbol for the same variable.

Definitions (D.1) to (D.10) of I, Section 2, subsection 1, will also hold for the functional calculus. Therefore, if A is a formula,

then ↑A and $\overset{\infty}{A}$ are also formulas; if A and B are formulas, then A∪B, A⊃B, A⊃⊂B, A∧B, A∨B, A→B, A⇒B, and A≡B are also formulas.

We now define three new quantifiers using the basic quantifier: (ϵ_x), $\exists_x$, $\forall_x$:

$$(D.11)\quad (\epsilon x)f(x) \overset{=}{_D} \neg(x)\neg f(x)$$

$$(D.12)\quad \exists x\ f(x) \overset{=}{_D} (\epsilon x)|\!-f(x)$$

$$(D.13)\quad \forall x\ f(x) \overset{=}{_D} (x)|\!-f(x)$$

Thus, if A is a formula containing x as a free variable, then (∈x)A, ∃xA and ∀xA are formulas.

(∈x)f(x) is read: "There exists at least one x with property f".
∃xf(x) is read: "The proposition f(x) is valid for at least one x".
∀xf(x) is read: "The proposition f(x) is valid for all x".

Because of the properties of (x) and (∈x) evident from the above axiom system, we call them the classical universal symbol and classical existential symbol, respectively.

The quantifiers ∀x and ∃x are called the nonclassical universal symbol and nonclassical existence symbol, respectively.

We now adopt the necessary conventions as regards subdivision of formulas by dots: the symbols ⊃⊂, ⊃, →, ↔, ≢, ≡, ∪, ∩, ∧, ∨, D predominate over the quantifiers; the symbols ¬, |∼, >, † preceding a quantifier act upon the entire subformula consisting of the quantifier and its scope. All other rules remain as before.

Thus, the expression

$$(x).\ f(x) \rightarrow g(x)$$

denotes the formula

$$(x)(f(x) \rightarrow g(x)),$$

the formula

$$(x).\ f(x) \cap g(x)\ .\rightarrow (x)\ h(x)$$

denotes

$$(x)(f(x) \cap g(x)) \rightarrow (x)\ h(x),$$

and

$$>(x).\ f(x) \rightarrow g(x)$$

denotes

$$>((x)(f(x) \rightarrow g(x))).$$

Finally, we shall simplify the formulas $\overline{(x)f(x)}$, $\overline{(\in x)f(x)}$, $\overline{\forall x f(x)}$, and $\overline{\exists x f(x)}$, replacing them by

$$(\bar{x})f(x),\ (\bar{\bar{\in}}\bar{\bar{x}})f(x),\ \bar{\bar{\forall}}xf(x),\ \bar{\bar{\exists}}xf(x).$$

SECTION 2. Axioms of the restricted functional calculus

We adopt three groups of axioms:

I: Any tautological formula of the propositional calculus is a provable formula.

II:

II.1) $(x) f(x) \to f(y)$,
II.2) $f(y) \to \exists x\, f(x)$,
II.3) $\neg(x)\, f(x) \to \exists x\, \neg f(x)$,
II.4) $\exists x\, \neg f(x) \to \neg(x)\, f(x)$,

III [In modern terminology, these "axioms" would be called rules of inference. (Tr.)]:

III.1) All axioms of II are provable formulas.

III.2) If A and B are provable formulas, then $A \wedge B$ is a provable formula.

III.3) If A and $A \to B$ are provable formulas, then B is a provable formula (principle of external deduction). Schematically:

$$\begin{array}{l} A \\ A \to B \\ \hline B. \end{array}$$

III.4) Principle of substitution: the following substitutions, carried out in a provable formula, yield a provable formula:

1) simultaneous substitution of the same expression for all occurrences of a propositional variable;

2) simultaneous substitution of the same expression, depending on variables x, y, ... (and perhaps also other variables), for all occurrences of a functional variable with arguments x, y, ...;

3) an object variable may be replaced throughout by another object variable or by the name of an object in the domain of values of the variable.

Of course, one should remember that

1) the principle of substitution applies only to free variables;

2) a variable appearing in the scope of a quantifier cannot be replaced by an expression depending on the quantified variable.

"Objects" in the restricted functional calculus are individuals which belong to a preassigned, suitably delineated domain.

III.5) Quantifier schema:

1) If $B(x)$ is an expression depending on x, A an expression not depending on x, and $A \to B(x)$ a provable formula, then $A \to (x)B(x)$ is also a provable formula.

2) If $B(x)$ is an expression depending on x, A an expression not depending on x, and $B(x) \to A$ is a provable formula, then $\exists x B(x)$ is also a provable formula,

SECTION 3. Some provable rules and formulas of the restricted functional calculus

Theorem VII. The following rule holds in the restricted functional calculus: If A and A⊃B are provable formulas, then B is a provable formula (principle of internal deduction). Schematically:

$$\frac{\begin{array}{l} A \\ A \supset B \end{array}}{B.}$$

Proof. Let A and A⊃B be provable formulas. Applying the principle of external deduction III+3) to the formula A⊃B and the provable formula

$$A \supset B \ .\rightarrow.\ A \rightarrow B$$

(see formula (28), I, Section 2, subsection 3), we get

$$\frac{\begin{array}{l} A \supset B \\ A \supset B \ .\rightarrow.\ A \rightarrow B \end{array}}{A \rightarrow B,}$$

i.e., A→B is a provable formula. Now, since the formula A is provable by assumption, another application of the principle of external deduction gives

$$\frac{\begin{array}{l} A \\ A \rightarrow B \end{array}}{B,}$$

i.e., B is a provable formula, Q.E.D.

Theorem VIII. The restricted functional calculus contains a subsystem isomorphic to the classical restricted functional calculus.

Proof. The restricted functional calculus contains formulas (1) to (12) of I, Section 2, subsection 3. Adding these formulas to the axioms of groups II and III, we clearly obtain an isomorphic image of the classical functional calculus; the universal symbol of the classical calculus corresponds to the quantifier (x) of our calculus, and the classical existence symbol to the quantifier ∃x of our calculus. This proves the theorem.

We shall retain the notation K+1 for the isomorphic image of the classical functional calculus whose existence we have just proved.

It is now easy to describe various classes of formulas which are provable in the restricted functional calculus.

Note the following:

1) Principle of generalization: Let A(x) be a provable formula containing x as a free variable; then the formula

(x)A(x)

is also provable.

2)

(44) $f(n) \to \exists x\, f(x)$

(where n is the name of an object belonging to the domain of the variable x)

(45) $(x).\ f(x) \to g(x)\ .\to.\ (x)\, f(x) \to (x)\, g(x)$.

The following formulas are not in K.1:

(46) $(\epsilon x)\, f(x) \to \exists x\, f(x)$

(47) $\neg(\epsilon x)\, f(x) \equiv (x)\neg f(x)$

(48) $\neg(x)\, f(x) \equiv (\epsilon x)\neg f(x)$

(49) $\exists x\, f(x) \wedge \neg f(\epsilon x)\, f(x) \to (\epsilon x)\, f(x)$.

We recall a further theorem:

Theorem IX. If $f(x) \equiv g(x)$ is a provable formula, then $(x)f(x) \equiv (x)g(x)$ is also a provable formula.

III

EXTENDED FUNCTIONAL CALCULUS AND ANALYSIS OF PARADOXES

SECTION 1. Extended functional calculus

To analyze the paradoxes of classical logic with the aid of the formal calculus developed above, we must be capable of constructing any classical formula in our new system. Now the restricted functional calculus is obviously inadequate for this purpose, and we therefore need an extension of the calculus. An extension of this type will be considered in this section.

First, using only certain elements of the system considered above, we construct a new system, which we shall call S·0. The first stage is the propositional calculus of S·0, which will include only two propositional functions, ¬a and a∧b, defined as in I, Section 2. Now introduce the definitions (D·1), (D·2), (D·3). In other words, we introduce the classical connectives but not the nonclassical ones. The concepts of formula and proposition are obviously more restricted than those of I, Section 2. Tautologies and contradictions are defined as before.

It is easily seen that no formula is provable in the propositional calculus of the system S·0.

We now construct the restricted functional calculus S·0. We proceed as in II, Section 1, up to the definition of formula. The latter concept is defined by the following rules:

1) Every propositional symbol (in S·1) is a formula;

2) Every function symbol in which the argument places are occupied by object names or object-variable symbols is a formula;

3) If A is a formula containing x as a free variable, then (x) A is a formula;

4) If A is a formula, then ¬A is a formula;

5) If A and B are formulas, then A∧B is a formula;

6) If a subformula of a formula is in the scope of a universal symbol, it cannot be in the scope of any other universal symbol for the same variable.

Now introduce definitions (D·1), (D·2), (D·3) and (D·11). Now, if A and B are formulas, then A∨B, A⊃B, A⊃⊂B are also formulas, and if A is a formula containing x as a free variable, then (∈x)A is also a formula.

The notation for formulas remains as before.

The only axioms we retain are I, III·2) and III·4), i.e., those involving classical formulas.

It is obvious that no formula is provable in the restricted functional calculus S.0.

We now extend the functional calculus S.0; this is done by adjoining all functions and propositions of the system S.0 to the set of objects, besides the original individuals. The object in axioms III.4) must also be interpreted in this extended sense. We are thus dealing with functions of functions and propositions, with the argument places of each function being referred to a definite domain of objects. An example of such a domain is the set of all propositions in the sense of S.0 or, say, the set of functions in the sense of S.0. We shall call the new system the full system S.0.

It is quite clear that the set of formulas available in the full system S.0 is exactly the same as in the unrestricted theory of types of the extended functional calculus of Hilbert and Ackermann (8), so that if the initial domain of individuals is the same in both systems, the variables are also the same.

It is obvious that the full system S.0 contains no provable formulas; the calculus only "discusses" formulas, so to speak.

We now extend the system S.0 as follows:

1) Introduce the nonclassical affirmation and negation of both functions and propositional variables, with the same properties as in I, Section 2, and then introduce all definitions (D.4) to (D.10) and (D.12), (D.13).

2) Correspondingly, extend the concepts of proposition and function. Of course, when this is done the concept of formula is also extended, but with one restriction which must be emphasized: Apart from individuals, the universe of objects contains only functions and propositions in the sense of the full system S.0. In other words, the domain of objects remains the same as that of the full system S.0.

In axioms I, III.2), III.4), the words "proposition", "function",, "formula" must be understood in the new, wider sense. However, the objects in part 3) of axiom III.4) are interpreted with an eye to the above restriction.

3) Introduce the axioms of group II, as well as axioms III.1), III.3), III.5), with the words "function", "proposition", "formula" understood in their new sense, as in the previous paragraph.

We call the new system S. Obviously, we must differentiate within the system S (and there is nothing to prevent us from so doing) between functional and propositional variables in the sense of S.0 and functional and propositional variables in the extended sense of S. Functions (propositions) in the sense of S.0 will be called simply functions (propositions) of classical logic; this is quite legitimate, in view of the relation between S_0 and the extended functional calculus of Hilbert-Ackermann. For the functional variables of classical logic we introduce the notation

$$f_k(\ ),\ g_k(\ ),\ \ldots,\ phi_k(\ ),\ psi_k(\ ),\ldots$$

The propositional variables of classical logic will be denoted by

$$a_k,\ b_k,\ c_k,\ \ldots$$

For the function and propositional variables of the system S we retain the notation of II, Section 1.

It is easy to prove that the system S cannot contain any expressions of classical logic equipotent to the formula $\uparrow a_k$. First we observe that, in view of Theorem IV (I, Section 2, subsection 2), it will suffice to show that any expression of classical logic of the form $(b_k)F(a_k,b_k)$ is meaningless if $\uparrow a_k$ is valid. But this is clear, for if $\uparrow a_k$ is valid, then a_k is meaningless; and then $F(a_k,b_k)$ must also be meaningless, so that the formula

$$\uparrow F(a_k,b_k)$$

is valid. Consequently, by axiom II_2),

$$\exists b_k\ \uparrow F(a_k,b_k)$$

and now axiom II_4) gives

$$\uparrow (b_k)\ F(a_k,b_k),$$

Q.E.D.

Of course, these arguments presuppose that the system S is consistent. The consistency of this system is an as yet unsolved problem, but all our attempts to obtain a contradiction have been unsuccessful, so that there is a considerable empirical basis for the assumption that S is consistent.

The system S will be necessary for our analysis of paradoxes in the extended functional calculus; it is the framework in which this analysis is carried out.

SECTION 2. Analysis of paradoxes in classical mathematical logic

1. General Remarks. The paradoxes of the classical extended functional calculus fall into two groups. The paradoxes of the first group are purely logical in character and their formulation requires no assumptions beyond the realm of logical formulas. Russell's paradox is an example. The paradoxes of the second group require the addition of certain formulas containing symbols for individual objects, functions or sentences. An example is Weyl's "heterological" paradox (9).

With regard to the first group of paradoxes, the system S is adequate to show that certain propositions are meaningless. By contrast, for the second group the results of our analysis will be based on premises of the type above, since the very formulation of the paradoxes in the classical system dictates their use.

We shall present an analysis of the paradoxes of Russell and Weyl. In this section a function of one variable will also be called a property. Wherever possible, we shall abbreviate the symbol phi( ) by phi.

2. Analysis of Russell's paradox. In the extended functional calculus of Hilbert and Ackermann, Russell's paradox arises when one considers the function

$$phi(phi),$$

which states that a class belongs to itself. Define

$$Pd(phi) =_{D} phi(phi).$$

By the provable formula of classical logic

$$a \supset_{c} a,$$

we can write

$$phi(phi) \supset_{c} phi(phi),$$

or, by the definition of the function Pd,

$$phi(phi) \supset_{c} Pd(phi).$$

The function ¬Pd belongs to the domain of values of the variable phi. Substituting ¬Pd for phi in the last formula, we get:

$$\neg Pd(\neg Pd) \supset_{c} Pd(\neg Pd).$$

This is Russell's paradox.

What happens in the system S? Consider the function

$$phi{\downarrow}k(phi{\downarrow}k)$$

and define

$$Pd(phi{\downarrow}k) =_{D} phi{\downarrow}k(phi{\downarrow}k).$$

Note that the domain of values of the variable phi↓k is the same as that of the variable phi introduced above for the classical version of Russell's paradox. But in the system S we cannot use the formula

$$a \supset_{c} a,$$

since it is not provable.

However, we do have the provable formula

$$a \equiv a \qquad (10)$$

Substituting phi↓k(phi↓k) for a in this formula, we get

$$phi{\downarrow}k(phi{\downarrow}k) \equiv phi{\downarrow}k(phi{\downarrow}k).$$

Now, by the definition of the function Pd,

$$Pd\ (phi{\downarrow}k) \equiv phi{\downarrow}k(phi{\downarrow}k).$$

The function ¬Pd belongs to the domain of values of the variable phi↓k. Substituting ¬Pd for phi↓k in the last formula, we get

$$(\alpha) \qquad Pd\ (\neg Pd) \equiv \neg Pd\ (\neg Pd)$$

The formula

$$a \equiv \neg a\ .{\equiv}\ {\uparrow}a \qquad (11)$$

is provable in S, and so we find

$$Pd\ (\neg Pd) \equiv \neg Pd\ (\neg Pd)\ .{\equiv}\ {\uparrow}Pd\ (\neg Pd),$$

and, by (α),

$${\uparrow}Pd\ (\neg Pd).$$

Now, by the provable formula

$${\uparrow}a \equiv {\uparrow}\neg a \qquad (12)$$

we get

$${\uparrow}\ \neg Pd\ (\neg Pd).$$

Thus, the proposition Pd(¬Pd) is meaningless, like its internal negation. The external negation of the proposition Pd(¬Pd) is false, as is its external affirmation.

3. Analysis of Weyl's paradox. We first carry out a formal reconstruction of Weyl's paradox in the classical extended functional calculus. The statement that the symbol z is heterological is expressed by a function H(z) defined as follows:

$$H(z) =_{D}. (\epsilon\ phi) . R(z,phi) \cap \neg phi(z) \qquad (13)$$

Here R(z,phi) is read "z designates phi". The domain of values of the variable is the set of symbols designating a property, and the domain of variables of phi is the set of properties. We adopt as an axiom the statement that the symbol "H" denotes the function H alone. Symbolically, this axiom is expressed by the following formulas:

1) $R(\text{"H"},H)$,

2) $R(\text{"H"},phi) \supset phi = H$.

Identity is defined in classical logic by the formula

$$x = y =_{D}. (f) . f(x) \supset f(y) \qquad (14)$$

Therefore, formula 2) can be rewritten as

2) $R(\text{"H"},phi) \supset. (f) . f(phi) \supset f(H)$.

It now follows from the definition of the function H that

(α) $H(\text{"H"}) \supset .(\epsilon\ phi). R(\text{"H"},phi) \cap \neg phi(\text{"H"})$.

By 2) and the provable formula

$$(f). f(phi) \supset f(H) .\supset. g(phi) \supset g(H)$$

we get

$$R(\text{"H"},phi) \supset. g(phi) \supset g(H).$$

Substituting the function $\neg phi(\text{"H"})$ for g(phi) in this formula, we get

$$R(\text{"H"},phi) \supset. \neg phi(\text{"H"}) \supset \neg H(\text{"H"}).$$

Hence we now deduce

$$R(\text{"H"},phi) \cap \neg phi(\text{"H"}) \supset. \neg phi(\text{"H"}) \cap. \neg phi(\text{"H"}) \supset \neg H(\text{"H"}).$$

In view of the formula

$$\neg phi(\text{"H"}) \cap. \neg phi(\text{"H"}) \supset \neg H(\text{"H"}) .\supset \neg H(\text{"H"}),$$

we get

$$R(\text{"H"},phi) \cap \neg phi(\text{"H"}) \supset \neg H(\text{"H"}).$$

Applying a well-known rule of the Hilbert-Ackermann functional calculus, we can write

$$(\epsilon\ phi). R(\text{"H"},phi) \cap \neg phi(\text{"H"}) .\supset \neg H(\text{"H"}).$$

In view of formula (α), we obtain

(A) $H(\text{"H"}) \supset \neg H(\text{"H"})$.

On the other hand, by the provable formula

$$f(n) \supset (\epsilon\ x)\ f(x),$$

where n is the name of an object belonging to the domain of values of the variable x, we get

$$R(\text{"H"},H) \cap \neg H(\text{"H"}) \supset. (\epsilon\ phi), R(\text{"H"},phi) \cap \neg phi(\text{"H"}),$$

or, by the definition of the function H,

$$R(\text{"H"},H) \cap \neg H(\text{"H"}) \supset H(\text{"H"}),$$

But since R("H",H) is an axiom, it follows that

(B) $$\neg H(\text{"H"}) \supset H(\text{"H"}).$$

Formulas (A) and (B) in combination give Weyl's paradox:

$$H(\text{"H"}) \supset\subset \neg H(\text{"H"}).$$

Now consider the situation in the system S. Since the argument is quite long, we shall omit references to the formulas of the propositional calculus used in the proofs. In each individual case it is easy to identify the formula being applied, and to verify its validity by constructing a truth table.

First and foremost, we must define the function H(z) in S:

$$H(z) \overset{D}{=}.\ (\epsilon\ \phi_k).\ R(z,\phi_k) \cap \neg\phi_k(z).$$

For $R(z,\phi_k)$, read: "z designates ϕ_k".

The domain of values of the variable z is the set of symbols of classical logic which designate properties, and the domain of values of ϕ_k is the set of properties considered in classical logic. Thus, the variables z and ϕ_k have the same respective domains of values as z and phi in the classical formulation of Weyl's paradox.

Formulas 1) and 2) now correspond to the formulas

$$1')\quad R(\text{"H"},H)$$

$$2')\quad R(\text{"H"},\phi_k) \rightarrow.\ (f_k).\ f_k(\phi_k) \leftrightarrow f_k(H) \qquad (15)$$

By the definition of the function H,

$$H(\text{"H"}) \rightarrow.\ (\epsilon\ \phi_k).\ R(\text{"H"},\phi_k) \cap \neg\phi_k(\text{"H"}).$$

Using axiom II↓1) (II, Section 2), we deduce from 2')

$$R(\text{"H"},\phi_k) \rightarrow.\ g_k(\phi_k) \leftrightarrow g_k(H).$$

Substituting $\neg\phi_k(\text{"H"})$ for $g_k(\phi_k)$ in this formula, we get

$$R(\text{"H"},\phi_k) \rightarrow.\ \neg\phi_k(\text{"H"}) \leftrightarrow \neg H(\text{"H"}).$$

Hence,

$$R(\text{"H"},\phi_k)\cap\neg\phi_k(\text{"H"}) \rightarrow.\ \neg\phi_k(\text{"H"}) \cap.\ \neg\phi_k(\text{"H"})\leftrightarrow\neg H(\text{"H"}).$$

Using the provable formula

$$\neg\phi_k(\text{"H"}) \cap.\ \neg\phi_k(\text{"H"}) \leftrightarrow \neg H(\text{"H"})\ .\rightarrow \neg H(\text{"H"}),$$

we get

$$R(\text{"H"},\phi_k) \cap \neg\phi_k(\text{"H"}) \rightarrow \neg H(\text{"H"}).$$

Now apply axiom III↓5 (II, Section 2) (quantifier schema) to formula (e): we get

$$\exists\ \phi_k\ .\ R(\text{"H"},\phi_k) \cap \neg\phi_k(\text{"H"})\ .\rightarrow \neg H(\text{"H"}).$$

Using formula (46) (II, Section 3), we now find

$$(\epsilon\ \phi_k).\ R(\text{"H"},\phi_k) \cap \neg\phi_k(\text{"H"})\ .\rightarrow \neg H(\text{"H"})$$

or, by the definition of the function H,

$$(A')\qquad H(\text{"H"}) \rightarrow \neg H(\text{"H"}).$$

On the other hand, by formula (44) (II, Section 3), we have

$$(1)\qquad R(\text{"H"},H) \cap \neg H(\text{"H"}) \rightarrow.\ \exists\ \phi_k\ .\ R(\text{"H"},\phi_k) \cap \neg\phi_k(\text{"H"}).$$

But by formula (49) (II, Section 3) and the definition of the function H,

$$\text{(II)} \quad \neg\vdash H(``H") \cap, \exists \varphi_k , R(``H", \varphi_k) \cap \neg\varphi_k(``H") ,\rightarrow H(``H"),$$

It follows from formula (II) that

$$\text{(III)} \quad \exists \varphi_k ,) R(``H", \varphi_k) \cap \neg\varphi_k(``H") , \rightarrow, \neg\vdash H(``H") \rightarrow H(``H"),$$

Formulas (I) and (III) give

$$R(``H", H) \cap \neg H(``H") \rightarrow, \neg\vdash H(``H") \rightarrow H(``H")$$

or

$$R(``H", H) \rightarrow: \neg H(``H") \rightarrow, \neg\vdash H(``H") \rightarrow H(``H")$$

Since the formula $R(``H", H)$ is an axiom, we get

$$\neg H(``H") \rightarrow, \neg\vdash H(``H") \rightarrow H(``H")$$

or, interchanging the premises,

$$\text{(B')} \quad \neg\vdash H(``H") \rightarrow, \neg H(``H") \rightarrow H(``H")$$

On the other hand, since formula (A') is provable, so is the formula

$$\text{(A'')} \quad \neg\vdash H(``H") \rightarrow, H(``H") \rightarrow \neg H(``H"),$$

Formulas (A") and (B') imply

$$\neg\vdash H(``H") \rightarrow, H(``H") \leftrightarrow \neg H(``H"),$$

Hence, by formula (33) (I, Section 2, subsection 3), we get the formula

$$\vdash H(``H")$$

and now, by formula (23) (I, section 2, subsection 3),

$$\vdash\neg H(``H"),$$

FOOTNOTES

1. The internal affirmation is considered to be identical with the proposition itself.

2. In this paper, "classical sentential calculus" will have a specific meaning: a truth-table calculus adequate for the sentential calculus of Hilbert-Ackermann (Grundzuge der theoretischen Logik) and the sentential calculus of PRINCIPIA MATHEMATICA (Whitehead and Russell).

3. The symbols ¬, |-, >, † act only on the letters and parentheses directly following them.

4. Here and below we shall assume that the classical sentential calculus employs the same symbols as those used here for classical formulas (in the sense defined above).

5. See footnote (4).

6. A. Heyting, Die formalen Regeln der intuitionistischen Logik, Sitz.-Ber. d. preuss. Akad. d. Wiss. (1930), pp. 42-56; A. Kilmogoroff, Zur Deutung der intuitionistischen Logik, Math. Z. 35 (1932), pp. 58-65.

7. Note that the symbol ∩ may also be replaced in K+1 by ∧ without replacing the symbol ↔ by ≡, but this transformation presents no special interest.

8. See Hilbert-Ackermann, Grundzuge der theoretischen Logik (1928), pp. 82-115.

9. On the distinction between those two types of paradoxes see: Hilbert and Ackermann, Grundzuge der theoretischen Logik (1928), p. 115; F. Ramsey, The Foundations of Mathematics, Proc. London Math. Soc., Ser. 2, Vol. 25, Part 5 (1926); R. Carnap, Abriss der Logistik (1929), p. 21; R. Carnap, Die Antionomien und die Unvollstandigkeit der Mathematik, Monatshefte f. Math. und physik (1934).

10. See I, Section 2, subsection 3, Theorem V.

11. See I, Section 2, subsection 3, formula (31).

12. See I, Section 2, subsection 3, formula (23).

13. See F. Ramsey, The Foundations of Mathematics.

14. See, e.g., R. Carnap, Abriss der Logistik, p. 15; also: Hilbert and Ackermann, Grundzuge der theoretischen Logik, p. 83.

15. This condition is even weaker than identity of the functions denoted by the symbol "H".

ON THE CONSISTENCY OF A THREE-VALUED CALCULUS

D.A. Bochvar

[From Matyematichyeski Sbornik (Recueil Mathematique), N.S. 12 (1943), pp. 353-369]

In our paper "A Three-Valued Logical Calculus and Its Application to the Analysis of paradoxes ..."(1) we described a certain three-valued system of mathematical logic, which we called S (2).

Within the system S one can formally prove that certain formulas of the classical extended functional calculus which lead to contradictions are meaningless.

Study of the system S is thus relevant for the problem of paradoxes.

Any study of the system S itself must naturally begin with the question as to whether one can establish its consistency as a whole, or, at least, the consistency of a fragment large enough to yield results sufficiently characteristic of those achievable in S.

In this paper we shall present certain results in this direction; some of them can be extended to a certain type of calculus based on the classical sentential calculus.

[Typist's note: Subscripts are indicated by ↓, and ↑ is used in place of the logical connective ↓ used in the original. The following lexicographic changes were also made: for propositional variables, we use lower case letters rather than capitals; for logical formulas, capitals rather than German capitals; for logical variables, lower case rather than German lower case; and for classical propositional calculus formulas, primed capitals rather than German capitals with superscript (Ø). S is used for capital Sigma, and w for lower case omega.]

SECTION 1

This section contains a brief resume of the axiom system of S. The basis for the system S is the propositional calculus (3).

Let a, b, c,... be propositional variables. Each of these variables can assume one of three truth values: T (read: "true"), F (read: "false"), and U (read: "meaningless").

The primitive propositional functions are: $\neg a$ (classical or internal (4) negation), $a \cap b$ (classical or internal conjunction) and $\vdash a$ (nonclassical or external affirmation). These are defined by truth tables:

a	¬a
T	F
F	T
U	U

a	b	a∩b
T	T	T
T	F	F
F	T	F
F	F	F
T	U	U
U	T	U
F	U	U
U	F	U
U	U	U

a	⊢a
T	T
F	F
U	F

Using the primitive functions, we define the classical functions:

(D.1) $a \cup b \underset{D}{=} \neg(\neg a \cap \neg b)$

(D.2) $a \supset b \underset{D}{=} \neg(a \cap \neg b)$

(D.3) $a \supset\subset b \underset{D}{=} (a \supset b) \cap (b \supset a)$

and the nonclassical functions:

(D.4) $a \wedge b \underset{D}{=} \vdash a \cap \vdash b$

(D.5) $a \vee b \underset{D}{=} \vdash a \cup \vdash b$

(D.6) $a \rightarrow b \underset{D}{=} \vdash a \rightarrow \vdash b$

(D.7) $a \leftrightarrow b \underset{D}{=} (a \rightarrow b) \cap (b \rightarrow a)$

(D.8) $a \equiv b \underset{D}{=} (a \leftrightarrow b) \cap (\neg a \leftrightarrow \neg b)$

(D.9) $>a \underset{D}{=} \vdash \neg a$

(D.10) $\bar{a} \underset{D}{=} \neg \vdash a$

(D.11) $\uparrow a \underset{D}{=} \neg(\vdash a \cap >a)$ (5)

The concept of formula is defined as usual. A formula is said to be provable in the propositional calculus if it takes the value T for all possible values of the variables. Formulas provable in the propositional calculus are also called tautologies. A formula which does not take the value T for any values of the variables is called a contradiction. A formula which contains, besides propositional variables, only symbols for classical functions, is called a classical formula of the propositional calculus.

The following theorems are valid:

1. No classical formula is provable in the propositional calculus.

2. No contradiction is provable in the propositional calculus.

3. A classical formula takes the value U whenever a propositional variable occurring in it takes the value U.

The propositional calculus serves as the basis for the restricted functional calculus. There are three kinds of variables:

1. Propositional variables a,b,c,...
2. Object variables x,y,z,...
3. Variables for functions of any finite number of object variables: f( ), g( ),..., phi( ), psi( ),...

The basic quantifier is the universal symbol (x). Formulas are defined as usual (6).

Using the basic quantifier, one defines new quantifiers (ϵx), ∃x, ∀x:

$$(D{\cdot}12)\quad (\epsilon\ x)\ f(x) \underset{D}{=} \neg(x)\neg f(x)$$

$$(D{\cdot}13)\quad \exists x\ f(x) \underset{D}{=} (\epsilon\ x)\ |\!-\ f(x)$$

$$(D{\cdot}14)\quad \forall x\ f(x) \underset{D}{=} (x)\ |\!-\ f(x) \qquad (7)$$

The usual restrictions are imposed on the use of the quantifiers (8). The quantifiers (x) and (ϵx) are called the classical universal and existential symbols, respectively, in view of their properties as defined by the axioms of the restricted calculus. The quantifiers ∀x, ∃x are called the nonclassical universal and existential symbols, respectively.

The following axioms [Tr: In modern terminology, some of these axioms would be called rules of inference. We shall continue to use the old terminology.] are adopted in the restricted functional calculus: (9)

I. Every tautological formula of the propositional calculus is a provable formula.

II. The following formulas are provable:

1. $(x)\ f(x) \rightarrow f(y)$ (10)
2. $f(y) \rightarrow \exists x\ f(x)$
3. $\neg(x)f(x) \rightarrow \exists x\ \neg f(x)$
4. $\exists x\ \neg f(x) \rightarrow \neg(x)\ f(x)$.

III.1. If A and $A \rightarrow B$ are provable formulas, then B is a provable formula. Schematically,

$$\frac{\begin{matrix} A \\ A \rightarrow B \end{matrix}}{B}.$$

III.2. Principle of substitution. The following substitutions, performed in a provable formula, yield a provable formula:

1) simultaneous substitution of the same formula for all occurrences of a propositional variable;
2) simultaneous substitution of the same formula, depending on variables x,y,... (and perhaps also other variables) for all occurrences of a functional variable with arguments x,y,...;
3) an object variable may be replaced throughout by another object variable or by the name of an object in the domain of values of the variable.

The principle of substitution applies only to free variables. Substitution of a symbol containing the variable bound by a quantifier for a variable in the scope of the quantifier is not allowed.

III.3. Quantifier schema.

1) If B(x) is a formula depending on x, A a formula not depending on x, and $A \rightarrow B(x)$ is a provable formula, then $A \rightarrow (x)B(x)$ is also a provable formula;

2) If B(x) is a formula depending on x, A a formula not depending on x, and $B(x) \rightarrow A$ is a provable formula, then $\exists x B(x) \rightarrow A$ is also a provable formula.

We now construct an auxiliary system -- S↓Ø (11).

We first construct the propositional calculus of the system S↓Ø. This construction utilizes only two (classical) propositional functions, ¬a and a∧b, defined as before, and then definitions (D↓1), (D↓2), (D↓3). Formulas are defined in the usual manner. Tautologies and contradictions are defined as before. It is easy to see that no formula is provable in the propositional calculus of S↓Ø.

Now, introducing symbols for function and object variables and the quantifier (x), with the correspondingly defined concept of formula, we construct the restricted functional calculus S↓Ø. The quantifier (∈x) is again defined by (D↓12). We adopt axiom I of the restricted calculus (reformulated for the system S↓Ø) and the reformulated substitution principle.

We now extend the functional calculus S↓Ø in precisely the same way as the classical restricted functional calculus is extended by constructing the extended calculus without the theory of types (12). The resulting system is called the full system S↓Ø. It is quite clear that the set of formulas available in the full system S↓Ø is precisely the same as that considered in the extended Hilbert-Ackermann functional calculus without theory of types. Obviously, the full system S↓Ø still contains no provable formulas. The system S is now obtained by combining the full system S↓Ø with the restricted functional calculus, using the following rules:

1. The universe of objects of the restricted functional calculus is now stipulated to be that of the full system S↓Ø. Thus, apart from individual variables, quantifiers may bound also variables for functions and propositions, though only in the sense of the full system S↓Ø.

2. There are two kinds of functional variables: functional variables in the sense of the full system S↓Ø, denoted by the symbols

f↓k( ), g↓k( ), ..., phi↓k( ), psi↓k( ), ...

and the functional variables of the restricted functional calculus, denoted by

f( ), g( ), ..., phi( ), psi( ),...

The functional variables of the second kind are thereby regarded as variables of a more general nature, in other words, functions in the sense of the full system S↓Ø may be substituted for functional variables of the restricted functional calculus, whereas functions not belonging to the full system S↓Ø may not be substituted for functional variables of the latter.

Thanks to this last condition, the functional variables of the restricted functional calculus now acquire the wider sense of functional variables in the system S as a whole (13).

Analogous conditions are imposed for the propositional variables, i.e., propositional variables in the sense of the full system $S{\downarrow}\emptyset$, similar to the conditions imposed on the functions with subscript k. Formulas are defined as for the restricted functional calculus, except that the words "proposition", "function" now denote propositions and functions in the sense of both the restricted functional calculus and the full system $S{\downarrow}\emptyset$.

Functions (propositions) in the sense of the full system $S{\downarrow}\emptyset$ will be called simply functions (propositions) of classical logic. We shall also speak of formulas and variables of classical logic, meaning formulas and variables in general in the sense of the full system $S{\downarrow}\emptyset$.

To abbreviate the meaning, formulas of classical logic will be denoted by capitals with subscript k, and variables of classical logic in general (irrespective of their nature) will sometimes be denoted (for brevity) by lower case letters with subscript k. Classical formulas of the propositional calculus (14) will be denoted by primed capitals.

SECTION 2

With an eye to a rigorous formulation of our problem, we shall first clarify the motives underlying our specific formulation. To this end, we first turn to the classical extended functional calculus without theory of types (in the sense of Hilbert-Ackermann). It is easily seen that the formulation of Russell's paradox in fact requires only a fragment of the calculus.

Russell's paradox can actually be derived in a more restricted axiom system, which we shall call AS (15):

1. Any tautological formula of the sentential calculus containing a single sentential variable A is provable,

2. If an expression phi(phi) is substituted for all occurrences of A throughout a provable formula, the result is a provable formula,

3. Define a functional constant F by

$$F(phi) \underset{D}{=} A(phi(phi))$$

(where A(phi(phi)) is a formula constructed from several expressions phi(phi) using the sentential connectives), Any formula obtained by substituting F for all occurrences of a functional variable throughout a provable formula is again provable.

In fact, define

(α) $$F(phi) \underset{D}{=} \neg phi(phi),$$

Since the formula

$$\neg phi(phi) \supset\subset \neg phi(phi)$$

is provable, we can use (α) to deduce that the formula

$$F(phi) \supset\subset \neg phi(phi)$$

is also provable. Substituting F for phi in this formula, we get Russell's adox:

$$F(F) \supset\subset \neg F(F),$$

Thus the axiom system AS contains all the formal prerequisites of Russell's paradox,

Turning now to the system S, we consider a subsystem which we shall call AS*, Formulas of AS* are defined as follows;

1. Any propositional variable symbol is a formula of AS*

2. Any quantifier-free formula of classical logic (in the sense of the full system S+Ø) is a formula of AS*,

3. If A is a formula of AS*, then $\neg A$, $|\neg A$, $>A$, $\circ A$, and $\bar{A}$ are formulas of AS*, If A,B are formulas of AS*, then $A \cap B$, $A \cup B$, $A \supset B$,

A⊃⊂B, A∧B, A∨B, A→B, A↔B, and A≡B are also formulas of AS*.

The axioms of AS* are:

1*. Any tautological formula of the propositional calculus is provable.

2*. If A↓k(v↓k↑1,...., v↓k↑n) is a formula of AS* belonging to the full system S↓Ø and depending only on the variables v↓k↑i (i=1,2,...,n) (these may be propositional or functional variables), define a constant function of classical logic by

F↓k(v↓k↑1,....,v↓k↑n) =_D A↓k(v↓k↑1,....,v↓k↑n).

In this equality the left and right hand sides may be interchanged in any formula (and this may be done either throughout the formula or only at isolated places) (16).

3*. Principle of substitution:

1) If a propositional variable (17) is replaced at all its occurrences in a provable formula by a formula of AS* (belonging to classical logic (18)), the result is a provable formula.

2) If a functional variable symbol of classical logic is replaced at all its occurrences in a provable formula by a formula of AS* (belonging to classical logic), which depends on the same arguments as the original variable (and possibly also on other arguments), the result is a provable formula.

3) If a variable is replaced at all its occurrences in a provable formula by another variable with the same domain of values or by a constant (as given by axiom 2* and belonging to the domain of variables of the original variable), the result is a provable formula.

4*. Principle of deduction:

If F and F→G are provable formulas, then G is a provable formula.

Comparing the axiom systems AS and AS*, it is natural to expect that were an analog of Russell's paradox deriveable in the system S this would be possible in the narrower axiom system AS*. Also relevant to a correct evaluation of the system AS* and its relation to S is the fact that the result concerning Russell's paradox (19) which can be proved in S remains valid in AS*.

We now present a proof of the consistency of the axiom system AS*. For brevity's sake we shall refer to it as the calculus AS*.

SECTION 3

In this section we shall prove that the calculus AS* is consistent.

A basis for the calculus AS* is a set of formulas containing:

1) every tautological formula of the propositional calculus;

2) every formula provable by applying substitution (axioms 2*, 3*) and definition of new constants (axiom 2*) to a tautological formula of the propositional calculus.

We shall denote the operation of substitution by Subst, the operation of definition by Def.

The concept of a maximal classical component of a propositional formula (20), which we now define, is essential for our argument.

A component of a formula F is any subformula G of F. If F contains G more than once as a subformula, we shall regard each "copy" of G appearing in F as a component of F. If G is a classical formula of the propositional calculus (see Section 1), we shall call it a classical component of F.

Any classical component of a formula F will be called a maximal classical component if it is not a component of any other classical component.

As an example, consider the formula

$$\neg\, a \cup b \rightarrow \neg \dagger a \cup \neg \dagger b.$$

The components of this formula are the formulas a, b, a, b, $\neg a$, $\dagger a$, $\dagger b$, $\neg \dagger a$, $\neg \dagger b$, $\neg\, a \cup b$, $\neg \dagger a \cup \neg \dagger b$, and $\neg\, a \cup b \rightarrow \neg \dagger a \cup \neg \dagger b$ (21). The classical components are a, b, a, b, $\neg a$, $\neg\, a \cup b$. The maximal classical components are $\neg a \cup b$, a, b (22).

Let F be a formula of the propositional calculus. Delete all its maximal classical components in succession, from left to right, and replace each of them by parentheses enclosing a numeral which counts the maximal classical components in order of deletion, from left to right. Denote the resulting symbol (which is clearly a certain operator of the propositional calculus) by [F]. We can now write the formula F as

$$(1) \qquad F = [F](A'_1, \ldots, A'_n)$$

where $A'_1, \ldots, A'_n$ are the maximal classical components of F (some of which may coincide) arranged in the order of the numerals assigned to them when they are deleted from F. It is obvious that if the form of the operator [F], all A'_i and their numerals are known, we can reconstruct the formula F uniquely. It is also obvious that if we have the formula A and observe the above order of operations in constructing (1), we can easily carry out this construction, and it is moreover unique.

Thus, if F is again the formula

$$\neg\, a \cup b \rightarrow \neg ?a \cup \neg ?b,$$

then

$$[F] = (1) \rightarrow \neg ?(2) \cup \neg ?(3) \qquad (23)$$

and the expression (1) is

$$F = [F](\neg\, a \cup b,\ a,\ b).$$

In the sequel we shall utilize the notation (1) for formulas of the propositional calculus, without further explanation.

We now prove two theorems which serve as the basis for our method.

Theorem 1. If

$$F = [F](A'_1, \ldots, A'_n)$$

is a tautological formula of the propositional calculus, then the formula

$$G = [F](B'_1, \ldots, B'_n)$$

obtained from F by replacing the maximal classical components A'_i by classical formulas B'_i of the propositional calculus (24) cannot be a contradiction, i.e., the formula

$$[F](B'_1, \ldots, B'_n)$$

cannot be a tautology.

PROOF. Assign the value U to all propositional variables appearing in the formulas in question (see Section 1). Then all the components A'_i and B'_i take the value U (25), and therefore both formulas F and G take the same value, T, since by assumption F is a tautology. But then, obviously, G cannot be a contradiction, and $\tilde{G}$ cannot be a tautology of the propositional calculus. Q.E.D.

Theorem II. If a basis of the calculus AS* contains formulas A and B, it must also contain the formula $A \cap B$.

PROOF. Suppose that a proof of the formula A consists of some combination of the operations Subst and Def, applied to a tautological formula F of the propositional calculus, while B is proved by similar operations on a tautology G. Obviously, the formula $F \cap G$ is a tautology of the propositional calculus. If necessary, rename the propositional variables of F and G in such a way that all substitution operations can now be applied in $F \cap G$ independently to the left and right of the connective $\cap$. It is now clear that, by applying to the formula $F \cap G$ the same combinations of Subst, Def (except for the names of the variables) as applied in the proofs of A and B, and then renaming the variables (if necessary), we get the required formula $A \cap B$. Q.E.D.

We now introduce some additional definitions. Given a formula F of the calculus AS*, we define an elementary component of F to be any subformula of F which does not have one of the forms $\neg A$,

$\vdash A$, $>A$, $\diamond A$, $\bar{A}$, $A \cap B$, $A \cup B$, $A \supset B$, $A \supset\subset B$, $A \wedge B$, $A \vee B$, $A \to B$, $A \leftrightarrow B$, $A \equiv B$ (26).

Associate a propositional variable with each elementary component of F, in such a way that different elementary components correspond to different propositional variables. Replacing each elementary component of F at all its occurrences by the corresponding propositional variable, we get a formula which we call a prototype of F, denoted by $\pi(F)$. Obviously, $\pi(F)$ is always a formula of the propositional calculus, i.e., it contains only propositional variables and connectives (27).

Call a formula F of the calculus AS* irregular if $\pi(F)$ is a contradiction of the propositional calculus, i.e., it never takes the value T.

We can now prove the following:

Theorem III. A basis of the calculus AS* contains no irregular formulas.

PROOF. Assume that a given basis contains an irregular formula, F say. Its proof starts with some tautology A of the propositional calculus, and proceeds by application of Subst and Def. Now it is clear that the only changes effected by these operations in the structure of the maximal classical components of A, and thereafter in the structure of the resulting formulas, are such that the maximal classical components become either formulas of classical logic or formulas which contain, apart from formulas of classical logic, only unchanged propositional variables. We may therefore state that if A has the form

$$[A](A'_1, \ldots, A'_n),$$

then the prototype of F must have the form

$$[A](B'_1, \ldots, B'_n).$$

Hence, by Theorem I, $\pi(F)$ cannot be a contradiction of the propositional calculus. However, this contradicts our assumption concerning F, and the proof is complete. Thus, in particular, a basis for the calculus AS* cannot contain formulas of the form

$$\bar{F}_k(F_k) \leftrightarrow F_k(F_k),$$

since their prototypes are

$$\bar{A} \leftrightarrow A,$$

which is a contradiction of the propositional calculus.

We now proceed to a consistency proof for the calculus AS*. To this end we need another definition.

Consider the proofs carried out in the calculus AS*.

We define a normal proof to be any proof in which the principle of deduction (axiom 4*, Section 2) is applied at most once, and then only at the end of the proof. Obviously, in a normal proof the principle of deduction is applied (if at all) to basis formulas. We now have the following theorem:

Theorem IV. Any provable formula A in the calculus AS* has a normal proof.

PROOF. The theorem is trivially true for basis formulas. The proof will obviously be complete if we prove the following two assertions:

a) If a formula G′ is obtained by the operations Subst and Def from a formula G which has a normal proof, then G′ also has a normal proof.

b) If a formula G is proved by applying the principle of deduction to formulas F and $F \to G$ which have normal proofs, then G also has a normal proof.

We first prove a). Suppose that G is proved via the schema

$$\frac{F \qquad F \to G}{G}$$

where, by assumption, F and $F \to G$ are basis formulas. Apply to $F \to G$ all Subst and Def operations needed to convert G to G′, and call the resulting formula $F' \to G'$. Apply to F all Subst and Def operations needed to convert F to F′. The resulting formulas F′ and $F' \to G'$ obviously belong to the basis.

Now the proof of G′ via the schema

$$\frac{F' \qquad F' \to G'}{G'}$$

is clearly normal, so that we have proved a).

We now prove b). Suppose that G is proved via the schema

$$\frac{F \qquad F \to G}{G}$$

By assumption, the basis contains formulas $F{\downarrow}1$ and $F{\downarrow}1 \to F$ from which the formula F is proved. By Theorem II it follows that the basis also contains the formula

$$F{\downarrow}1 \cap (F{\downarrow}1 \to F).$$

Analogous reasoning shows that the basis contains formulas

$$F{\downarrow}2,\ F{\downarrow}2 \to (F \to G),\ F{\downarrow}2 \cap (F{\downarrow}2 \to (F \to G)).$$

Again by Theorem II, we see that the basis also contains the formula

$$(1) \qquad F{\downarrow}1 \cap (F{\downarrow}1 \to F) \cap F{\downarrow}2 \cap (F{\downarrow}2 \to (F \to G)),$$

We now use the following tautology of the propositional calculus:

$$(\alpha) \qquad a \cap (a \rightarrow b) \cap c \cap (c \rightarrow (b \rightarrow d)) \rightarrow d.$$

Replace the variables a,b,c,d by the formulas $\pi(F{\downarrow}1)$, $\pi(F)$, $\pi(F{\downarrow}2)$, $\pi(G)$, respectively. Obviously the resulting formula

$$(\beta) \qquad \pi(F{\downarrow}1) \cap (\pi(F{\downarrow}1) \rightarrow \pi(F)) \cap \pi(F{\downarrow}2) \cap (\pi(F{\downarrow}2) \rightarrow (\pi(F) \rightarrow \pi(G))) \rightarrow \pi(G)$$

is also a tautology, and is therefore contained in the basis. In deriving (β) from (α), we ensure that the propositional variables in all prototypes are so chosen that the substitutions converting $\pi(F{\downarrow}1)$, $\pi(F)$, $\pi(F{\downarrow}2)$, $\pi(G)$ to $F{\downarrow}1$, F, $F{\downarrow}2$, G, respectively, can be performed independently in (β). Performing these substitutions, we clearly get the formula

$$(2) \qquad F{\downarrow}1 \cap (F{\downarrow}1 \rightarrow F) \cap F{\downarrow}2 \cap (F{\downarrow}2 \rightarrow (F \rightarrow G)) \rightarrow G, \qquad (28)$$

which is in the basis. With the formulas (1) and (2), we can now construct the required normal proof via the schema

$$\frac{\begin{array}{l} F{\downarrow}1 \cap (F{\downarrow}1 \rightarrow F) \cap F{\downarrow}2 \cap (F{\downarrow}2 \rightarrow (F \rightarrow G)) \\ F{\downarrow}1 \cap (F{\downarrow}1 \rightarrow F) \cap F{\downarrow}2 \cap (F{\downarrow}2 \rightarrow (F \rightarrow G)) \rightarrow G. \end{array}}{G.}$$

This proves b), and thereby Theorem IV.

Theorem V. No irregular formula is provable in the calculus AS*.

PROOF. Assume that an irregular formula A is provable in the calculus AS*. Then by Theorem IV, there is a normal proof of A. Now this normal proof obviously starts with a basis formula and ends with an application of the deduction principle:

$$\frac{\begin{array}{l} B \\ B \rightarrow A \end{array}}{A}$$

where B, B→A are basis formulas. By assumption, the prototype $\pi(A)$ is a contradiction of the propositional calculus. By Theorem III, $\pi(B \rightarrow A)$ cannot be a contradiction. Now $\pi(B \rightarrow A)$ is the same as $\pi(B) \rightarrow \pi(A)$ (29). Consequently, we can find an assignment of truth values (30) for $\pi(B) \rightarrow \pi(A)$ such that

$$\pi(B) \rightarrow \pi(A) = T.$$

Now, obviously, since $\pi(A)$ is a contradiction, this assignment of truth values must make

$$\pi(B) = F \text{ or } \pi(A) = T.$$

Hence the formula

$$\pi(B) \cap (\pi(B) \rightarrow \pi(A))$$

is always a contradiction of the propositional calculus. But this formula is a prototype of the formula

$$B \cap (B \rightarrow A),$$

which is in the basis, by Theorem II. This implies that the basis contains an irregular formula, contradicting Theorem III. Q.E.D.

An obvious corollary of Theorem V is:

Theorem VI. The calculus AS* is consistent.

SECTION 4

This section is devoted to several remarks on the above consistency proof.

The constructive (finitary) character of this proof is self-evident. Without essential changes in the proof, certain restrictive conditions in the axioms of AS* can be eliminated.

Axiom 2* permits introduction of definitions only for constant functions which are functions of classical logic. This restriction is quite natural, for in general there is no interest in considering in S constant functions defined by formulas of nonclassical logic. Moreover, it does not in fact weaken the result, for such functions do not belong to the universe of objects of the system S, and so may always be easily eliminated from all proofs.

Axiom 3* permits substitution only of formulas in the sense of AS* which are formulas of classical logic. This restriction is also inessential. For any substitution of formulas of AS which involve nonclassical connectives may always be reduced (31) to substitutions of the permitted variety. This can be achieved by suitable choice of the initial tautologies of the propositional calculus, and this choice in turn may be effected by the method used in the proof of Theorem IV. The required reduction is made possible by the fact, that, by axiom 1*, any tautological formula of the propositional calculus is provable in AS*. The condition that the calculus AS* contains no functional variables without subscript k (i.e. functional variables in the general sense of the system S) is also inessential, for the previous remark easily shows that their introduction has no effect on the structure or the nature of the proof.

Thus we see that all the restrictions adopted above have a single purpose -- to simplify the arguments without essentially weakening the result.

It follows that the consistency proof presented in Section 3 may be regarded as a consistency proof for the system S without quantifiers and the relevant axioms and rules. It is interesting that some of our results may be extended to other formal systems of a certain, so to speak elementary, structure. We refer here to extended functional calculi based on the classical sentential calculus in the same way as the calculus AS* is based on the propositional calculus. (32)

Consider a calculus of this type, in which the admissible types of variables have been described and the formulas defined. Assume, moreover, that the axioms of this calculus make every tautological formula of the sentential calculus provable, introduce the principle of deduction in its usual form, and indicate the admissible types of definition and substitution.

Call a calculus of this kind an elementary extension of the functional calculus.

Under these assumptions, the concepts of basis, prototype, irregular formula and normal proof are defined exactly as for the calculus AS*. Hence it is easy to prove the following theorem:

Theorem VII. A necessary and sufficient condition for consistency of an elementary extension of the functional calculus is that no basis contain irregular formulas.

Thus, for example, the elementary calculus derived from the system AS (see beginning of Section 2) by adjoining the principle of deduction is inconsistent, since its basis contains an irregular formula (Russell's paradox).

We now briefly show how the above consistency proof will yield a nonfinitary consistency proof for the entire system S. Only the main lines of the proof will be indicated.

We first construct a new, nonfinitarily formulated calculus, which we call AS^*_w. To this end the propositional calculus is enriched by admitting countable conjunctions and disjunctions (both classical and nonclassical), defined as follows:

1) The classical countable conjunction

$$a_1 \cap a_2 \cap \dots \cap a_n \cap \dots$$

has the value U if for some i ($i = 1,2,\dots,n,\dots$)

$$a_i = U\,;$$

it has the value F if for all i ($i = 1,2,\dots,n,\dots$)

$$a_i \neq U\,,$$

but for at least one i ($i=1,2,\dots,n,\dots$)

$$a_i = F\,;$$

and the value T if for all i ($i=1,2,\dots,n,\dots$)

$$a_i = T.$$

2) $a_1 \cup \dots \cup a_n \cup \dots =_D \neg(\neg a_1 \cap \dots \cap \neg a_n \cap \dots)$.

3) $a_1 \wedge \dots \wedge a_n \wedge \dots =_D \models a_1 \cap \dots \cap \models a_n \cap \dots$.

4) $a_1 \vee \dots \vee a_n \vee \dots =_D \models a_1 \cup \dots \cup \models a_n \cup \dots$.

Tautologies, contradictions and classical formulas are defined as before.

Because of the introduction of countable conjunctions and disjunctions, the concept of formula in the sense of AS* and the axioms 1*, 2*, 3* are now replaced by formula in the sense of AS^*_w and axioms 1^*_w, 2^*_w, 3^*_w, respectively. The principle of deduction is of course unchanged. As before, we define the concepts of basis, maximal classical component (33), elementary component, prototype, irregular formula and normal proof. Using these tools one can now prove the consistency of the calculus AS^{*}_{w}. The proof proceeds formally as before, but now the arguments are no longer finitary.

Now observe that the set of constants introduced by definitions in the system S (34) is countable. Let these be

$$a_1,\ a_2,\ \ldots,\ a_n,\ \ldots$$

This being so, any proof in the system S can be converted into a proof in the system AS^{*}_{w}, by reinterpreting the formulas $(x)f(x)$, $(Ex)f(x)$, $\forall x\, f(x)$, $\exists x\, f(x)$, respectively, as follows:

$$f(a_{i_1}) \cap \ldots \cap f(a_{i_n}) \cap \ldots$$
$$f(a_{i_1}) \cup \ldots \cup f(a_{i_n}) \cup \ldots$$
$$f(a_{i_1}) \wedge \ldots \wedge f(a_{i_n}) \wedge \ldots$$
$$f(a_{i_1}) \vee \ldots \vee f(a_{i_n}) \vee \ldots$$

where $a_{i_1}, \ldots, a_{i_n}, \ldots$ is the domain of values of the variable x. When this is done, the axioms and rules for quantifiers in S become provable formulas and derived rules of the system AS^{*}_{w} (35) This transformation converts any contradictory formula of S into a contradictory formula of AS^{*}_{w}. Since AS^{*}_{w} is consistent, it now follows that the system S is consistent.

18. I.e., a formula of classical logic in the sense of Section 1.

19. loc. cit., pp. 304-305.

20. By a propositional formula we mean a formula containing only propositional variables and propositional connectives.

21. It is convenient to regard the entire formula as a component of itself, in view of the possibility that the formula itself is classical and is therefore its own (unique) maximal classical component.

22. Obviously, one copy of a formula G may occur in a formula F as a maximal classical component, while another copy of the same formula G occurs in F as a nonmaximal classical component.

23. Note that even if F and G are different formulas, it may happen that [F]=[G]; thus, if the formulas F and G are identical so are the symbols [F] and [G], but the converse need not be true.

24. B'↓l need not contain the same variables as A'↓l. It is clear that B'↓k will be the maximal classical components of the formula G.

25. It follows immediately from the truth table of the functions ¬A, A∩B that a classical propositional formula takes the value S whenever at least one of is arguments takes this value.

26. Thus, if

$$F{\downarrow}k(phi{\downarrow}k) \underset{D}{=} A{\downarrow}k(phi{\downarrow}k),$$

then F↓k(phi↓k) is clearly an elementary component of any formula of which it is a subformula. But A↓k(phi↓k) cannot be an elementary component of any formula of which it is a subformula.

27. The prototype of a formula F is clearly unique up to the names of its propositional variables. If necessary, this remaining ambiguity can be removed by fixing the propositional variables in a given proof once for all, so that each elementary component appearing in the proof will correspond throughout the proof to the same propositional variable.

28. Direct substitution of the formulas F↓1, F, F↓2, G for the variables A,B,C,D, in (α) is not permitted by the axioms of AS*, since they need not be formulas of classical logic. However, this restriction imposed in AS* by the axioms for substitution, is not essential (see below, Section 4).

29. See footnote (27).

30. See Section 1.

31. In the sense that the final result is the same formula.

32. I.e., not containing quantifiers and the relevant axioms and rules.

33. The construction of the symbol [F] must now involve also transfinite numbers of the second number class for enumeration of the maximal classical components.

34. Of course, there are those constants introduced by definitions in the classical extended functional calculus without theory of types (which is inconsistent).

35. Axioms containing a free variable must first be prefixed by the corresponding (classical) universal symbol, and then the axioms in the new interpretation become basis formulas of the calculus AS^*_W. Axiom III_3 (see Section 1) corresponds in AS^*_W to a derived (i.e., deducible from the axioms of AS^*_W) rule, consisting of the following two assertions:

1. Given a sequence of formulas $B_1, \ldots, B_n, \ldots$ if the formula $A \to B_n$ is provable for each n, then so is the formula
$$A \to B_1 \cap \ldots \cap B_n \cap \ldots$$

2. Given a sequence of formulas $B_1, \ldots, B_n, \ldots$ if the formula $B_n \to A$ is provable for each n, then so is the formula
$$B_1 \vee \ldots \vee B_n \vee \ldots \to A.$$

Review of Bochvar's "On a three-valued logical calculus..." by Alonzo Church, from the Journal of Symbolic Logic, 4,2 (June 1939), p. 99.

The author employs a three-valued propositional calculus whose character may be indicated by the following typical truth-tables: [... Church gives at this point the truth tables for

¬, |¬, >, †, $\bar{a}$, ∩, ⊃⊂, ↔, and ≡ ...] On this basis he develops a (three-valued) system of logic, introducing first a functional calculus of first order, and then an extended functional calculus -- analogously to the treatment of Hilbert-Ackermann (365,1), but without a theory of types.

This system is used for an "analysis" of the paradoxes of Russell and Grelling, these paradoxes being thought of as taken from a two-valued system and therefore expressed in terms of the negation ¬. If Q is the formula which leads to the Russell paradox in a two-valued system, by means of the equivalence Q ⊃⊂ ¬Q, then, in the three-valued system, Q ≡ ¬Q is demonstrable but, insted of leading to paradox, leads only to †Q.

The author overlooks that the three-valued system is itself inconsistent through the presence in it of another form of the Russell paradox, in which the negation $\bar{a}$ appears instead of ¬a.

Additional comment by Church in Journal of Symbolic Logic review, 5,3 (September 1940), p. 119.

The reviewer would take this opportunity to correct an error made in a review of [Bochvar's "On a three-valued ... "]. In that paper the author does not propose an unrestricted three-valued logic without a theory of types. Insstead, he first introduces an auxiliary system (extended functional calculus) S+Ø, which has no rule of types, but which employs as propositional connectives only anb an ¬a and connectives definable in terms of these (see truth-tables in the review referred to). Then he extends his three-valued functional calculus of the first order by allowing formulas of S+Ø to appear in place of the free individual variables and (propositional or functional) variables of S+Ø to appear in the place of bound individual variables. The resulting system does not have the immediate inconsistency which the reviewer charged. On the contrary, the suggested alternative to the theory of types is far from devoid of interest. The major question, it would seem, is not that of consistency, but whether it is possible to obtain along these lines a system adequate to the purposes for which the extended functional calculus is usually employed, e.g., to the theory of finite cardinal numbers or to analysis -- Bochvar does not discuss this point.