

DEGREES AND MATCHINGS

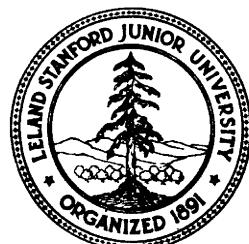
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STAN-CS-72-268

MARCH 1972

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DEGREES AND MATCHINGS

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Abstract

Let n , b , d be positive integers. D. Hanson proposed to evaluate $f(n, b, d)$, the largest possible number of edges in a graph with n vertices having no vertex of degree greater than d and no set of more than b independent edges. Using the alternating path method, he found partial results in this direction. We complete Hanson's work; our proof technique has a linear programming flavor and uses Berge's matching formula.

AMS(MOS) subject classification numbers (1970): Primary 05C35
Secondary 90C10

Keywords and phrases: Degrees, matchings, independent edges,
Berge's matching formula.

This research was supported by the National Science Foundation under grant number GJ-992, and the Office of Naval Research under grant number N-00014-67-A-0112-0057 NR 044-402. Reproduction in whole or in part is permitted for any purpose of the United States Government.

1. Introduction

Erdős and Rado [5] proved that given any positive integers n, k there is always an integer a with the following property: if F is any family of more than a sets, each of cardinality n , then some k members of F have **pairwise** the same intersection. Let us denote the smallest such a by $\varphi(n, k)$. Some results on $\varphi(n, k)$ can be found in [5], [1] and [3]. Obviously, $\varphi(2, k)$ is the maximum number of edges in a graph containing no vertex of degree greater than $k-1$ and no set of more than $k-1$ independent edges. The values of $\varphi(2, k)$ have been **determined** by N. Sauer (to appear):

$$\varphi(2, k) = \begin{cases} k(k-1) & \text{if } k \text{ is odd,} \\ (k-1)^2 + \frac{1}{2}k - 1 & \text{if } k \text{ is even.} \end{cases} \quad (1)$$

D. Hanson [6] considered a little more general problem. By an (n, b, d) -graph we shall mean a graph G such that

- (i) G has n vertices,
- (ii) G contains no set of more than b independent edges,
- (iii) G contains no vertex of degree greater than d .

The largest possible number of edges of an (n, b, d) -graph will be denoted by $f(n, b, d)$. In the Greek alphabet notation of [7], $f(n, b, d)$ is the maximum of $q(G)$ subject to the constraints

$$p(G) = n, \quad \beta_1(G) \leq b, \quad \Delta(G) \leq d.$$

Obviously, $f(n, b, d) = f(n, b, n-1)$ whenever $d > n-1$. Similarly, $f(n, b, d) = f(2b+1, b, d)$ whenever $n < 2b+1$. Hence we can restrict ourselves to the case $n > d+1, n \geq 2b+1$.

Apart from the most difficult case (d odd and $< 2b$, n small), the values of $f(n, b, d)$ have already been obtained by Hanson [6].

His proof technique is based on the alternating path method. We will adopt a different approach, related to linear programming. This technique simplifies the proofs and enables us to complete the evaluation of $f(n, b, d)$ without much additional effort. The result goes as follows.

THEOREM. Let n, b, d be positive integers with $n > 2b+1$.

A. If $d \leq 2b$ and $n < 2b + \left\lceil \frac{b}{\lceil \frac{d+1}{2} \rceil} \right\rceil$ then

$$f(n, b, d) = \begin{cases} \min\left\{\left\lfloor \frac{nd}{2} \right\rfloor, bd + \left\lceil \frac{2(n-b)}{d+3} \right\rceil \cdot \frac{d-1}{2}\right\} & \text{if } d \text{ is odd,} \\ \frac{nd}{2} & \text{if } d \text{ is even.} \end{cases}$$

B. If $d \leq 2b$ and $n > 2b + \left\lceil \frac{b}{\lceil \frac{d+1}{2} \rceil} \right\rceil$ then

$$f(n, b, d) = bd + \left\lceil \frac{b}{\lceil \frac{d+1}{2} \rceil} \right\rceil \cdot \left\lceil \frac{d}{2} \right\rceil$$

c. If $d \geq 2b+1$ then

$$f(n, b, d) = \begin{cases} \max\left\{\left\lfloor \frac{2b+1}{2} \right\rfloor, \left\lfloor \frac{b(n+d-b)}{2} \right\rfloor\right\} & \text{if } n \leq b+d, \\ bd & \text{if } n > b+d. \end{cases}$$

In proving that $f(n, b, d)$ cannot exceed the values given by our Theorem, we shall make use of Berge's matching formula [2]

$$\beta_1(G) = \min \frac{1}{2} (p(G) + |S| - k_0(G-S)) . \quad (2)$$

Here $\beta_1(G)$ is the maximum number of independent edges in G ,
 $P(G)$ is the number of vertices of G , S runs through all the subsets
of the vertex-set of G and finally, $k_0(G-S)$ is the number of odd
components (i.e., components with odd number of vertices) of the
 S -deleted graph $G-S$.

On the other hand, we shall construct (n, b, d) -graphs having
 $f(n, b, d)$ edges. Then we shall use the following simple proposition:
given any nonnegative integers n_1, n_2, d with $1 < d < n_1+n_2$ and
 $(d-1)n_1 + dn_2$ even, there is a graph G with n_1+n_2 vertices, n_1 of
them of degree $d-1$ and the remaining n_2 of degree d . Actually,
this statement is a corollary of a general existence theorem due to
Erdős and Gallai [4]: Let $d_1 \geq d_2 > \dots \geq d_n$ be nonnegative integers.
A necessary and sufficient condition for the existence of a graph G
with n vertices u_1, u_2, \dots, u_n , each u_i of degree d_i , is that
 $\sum_{i=1}^n d_i$ be even and

$$\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min\{d_i, k\}$$

for each $k = 1, 2, \dots, n-1$.

We conclude this section with two observations made by Hanson [6].
Firstly, Sauer's formula (1) appears to be a corollary of the theorem.
Indeed, one has

$$\varphi(2, k) = \max_n f(n, k-1, k-1) = \lim_{n \rightarrow \infty} f(n, k-1, k-1) = (k-1)^2 + \left[\frac{k-1}{\frac{k}{2}} \right] \cdot \left[\frac{k-1}{2} \right] .$$

Similarly, the theorem implies that a graph with n vertices and at
most b independent edges can have at most

$$f(n, b, n-1) = \max\left\{\binom{2b+1}{2}, b(n-b) + \binom{b}{2}\right\}$$

edges. This has been proved by Moon [8]. As noticed by J. A. Bondy, Moon's result follows instantly from Berge's matching formula (2).

2. Upper Bounds

LEMMA 1.

$$f(n, b, d) \leq \max \left(\min \left\{ dn_0, \left[\frac{n_0(d+n-n_0)}{2} \right] \right\} + \sum_{i=1}^m \min \left\{ \binom{n_i}{2}, \left[\frac{dn_i}{2} \right] \right\} \right)$$

where the maximum runs over all partitions

$$n = n_0 + n_1 + n_2 + \dots + n_m$$

into nonnegative integers with $m = n - n_0 - 2b$ and all n_i ($1 \leq i \leq m$) odd.

Proof. Let $G = (V, E)$ be an arbitrary (n, b, d) -graph. By Berge's formula (2), there is a set $S \subset V$ with $k_0(G-S) \geq n + |S| - 2b$. Let the odd components of G be G_1, G_2, \dots, G_M . Then $M > m = n + |S| - 2b$. Let us denote $|S|$ by n_0 and the number of vertices of each G_i ($1 \leq i < m$) by n_i ; let us also set

$$n_m = n - \sum_{i=0}^{m-1} n_i.$$

Then n_m has the parity of $n - n_0 - (m-1) = 2b - 2n_0 + 1$ and so all n_i 's with $1 \leq i \leq m$ are odd. We denote by x the number of edges of G having both endpoints in S , by y the number of edges of G having exactly

one endpoint in S . For each $i = 1, 2, \dots, m-1$ we denote by z_i the number of edges of G_i and finally we denote by z_m the number of the remaining edges in G . Obviously, we have

$$2x + y \leq dn_0 \quad (3)$$

$$y \leq n_0(n - n_0) \quad (4)$$

$$z_i \leq \min \left\{ \binom{n_i}{2}, \left[\frac{dn_i}{2} \right] \right\} \quad (1 \leq i \leq m) \quad . \quad (5)$$

Summing (3) and (4) and using the integrality of $x+y$ we obtain

$$x+y \leq \left[\frac{n_0(d+n-n_0)}{2} \right] . \quad (6)$$

Besides, (3) itself implies

$$x+y \leq dn . \quad (7)$$

Now, the desired conclusion follows from (5), (6), (7) and the fact that G has exactly $x+y+z_1+z_2+\dots+z_m$ edges.

LEMMA 2.

$$f(n, b, d) \leq bd + \left[\frac{b}{d+1} \right] \cdot \left[\frac{d}{2} \right] . \quad (8)$$

(In particular, $f(n, b, d) \leq bd$ whenever $d \geq 2b+1$.) Besides, if d is odd then

$$f(n, b, d) < bd + \left[\frac{2(n-b)}{d+3} \right] y . \quad (9)$$

Proof: Let n, b, d be given. For each positive integer s , we set

$$g(s) = \min\left\{\binom{s}{2}, \left[\frac{ds}{2}\right]\right\} = \begin{cases} \binom{s}{2} & \text{if } s \leq d+1, \\ \left[\frac{ds}{2}\right] & \text{if } s \geq d+1. \end{cases}$$

To each partition

$$n = n_0 + n_1 + \dots + n_m \quad (10)$$

with $n_1 \geq n_2 \geq \dots \geq n_m$ and all the n_i 's ($i = 1, 2, \dots, m$) odd, we assign a positive integer -- the **smallest** subscript $k \geq 1$ such that $n_i = 1$ for all $i > k$. Among all the partitions (10) which maximize

$$\min \left\{ dn_0, \left[\frac{n_0(d+n-n_0)}{2} \right] + \sum_{i=1}^m g(n_i) \right\}, \quad (11)$$

we choose one with minimum k .

If $k > 1$ then necessarily $n_i \geq d+1$ for all i with $1 \leq i \leq k$. Indeed, it is not difficult to check that

$$s \leq d \Rightarrow g(s) + g(t) \leq g(s+t-1).$$

Now, if $n_k \leq d$ then set $n_k^* = 1$, $n_{k-1}^* = n_{k-1} + n_k - 1$ and $n_i^* = n_i$ ($i \neq k-1, k$). Then

$$\sum_{i=1}^m g(n_i) \leq \sum_{i=1}^m g(n_i^*)$$

and so the partition $n = n_0^* + n_1^* + \dots + n_m^*$ maximizes (11). However, we have

$$|\{i: i > 1, n_i^* = 1\}| > |\{i: i > 1, n_i = 1\}|$$

contradicting the **minimality** of k .

Now, we shall distinguish three cases.

Case 1. $n_1 \leq d$. Then necessarily $k = 1$ and so $n_1 = n - n_0 - (m-1) = 2b - 2n_0 + 1$. Since $1 \leq n_1 < d$, we have

$$b - \frac{d-1}{2} \leq n_0 \leq b \quad . \quad (12)$$

Lemma 1 yields

$$f(n, b, d) \leq dn_0 + \sum_{i=1}^m g(n_i) = dn_0 + \binom{n_1}{2} = dn_0 + \binom{2b - 2n_0 + 1}{2} .$$

Since $F(n_0) = dn_0 + \binom{2b - 2n_0 + 1}{2}$ is a convex function with

$F(b - \frac{d-1}{2}) = F(b) = bd$ and n_0 satisfies the constraints (12), we have $f(n, b, d) \leq bd$. Hence in this case both inequalities (8), (9) are satisfied.

Case 2. $n_1 > d+1$, d even. Here Lemma 1 gives

$$\begin{aligned} f(n, b, d) &< dn_0 + \sum_{i=1}^m g(n_i) = dn_0 + \sum_{i=1}^k \frac{dn_i}{2} = dn_0 + \frac{d}{2} \sum_{i=1}^k n_i = \\ &= dn_0 + \frac{d}{2} (n - n_0 - (m-k)) = bd + k \end{aligned}$$

Besides, we have $k(d+1) < \sum_{i=1}^k n_i = n - n_0 - (m-k) = 2b + k - 2n_0 > 2b + k$

and so $k \leq [\frac{2b}{d}]$. But then

$$f(n, b, d) \leq bd + k \cdot \frac{d}{2} \leq bd + [\frac{2b}{d}] \cdot \frac{d}{2}$$

which is the desired inequality (8).

Case 3. $n_1 \geq d+1$, d odd. Again, Lemma 1 yields

$$\begin{aligned} f(n, b, d) &\leq dn_0 + \sum_{i=1}^m g(n_i) = dn_0 + \sum_{i=1}^k \frac{dn_i - 1}{2} = dn_0 + \frac{d}{2} \sum_{i=1}^k n_i - \frac{k}{2} = \\ &= dn_0 + \frac{d}{2} (n - n_0 - (m-k)) - \frac{k}{2} = bd + k \cdot \frac{d-1}{2} \end{aligned}$$

We have $n_i \geq d+1$ whenever $1 < i < k$. Moreover, each n_i ($1 \leq i \leq k$) is odd while $d+1$ is even. Hence we have $n_i \geq d+2$ whenever $1 \leq i \leq k$.

Besides, we have $k(d+2) < \sum_{i=1}^k n_i = n - n_0 - (m-k) = 2b - 2n_0 + k$ and so

$$k \leq \left[\frac{2b - 2n_0}{d+1} \right]$$

If $n_0 > 2b - n + \frac{2n-2b}{d+3}$ then

$$k \leq \left[\frac{2b - 2n_0}{d+1} \right] \leq \left[\frac{2n-2b}{d+3} \right] ;$$

if $n_0 \leq 2b - n + \frac{2n-2b}{d+3}$ then

$$k \leq n + n_0 - 2b \leq \left[\frac{2n-2b}{d+3} \right] .$$

Moreover, since $n_0 \geq 0$, we have $k \leq \left[\frac{2b - 2n_0}{d+1} \right] \leq \left[\frac{2b}{d+1} \right]$. The inequalities

$$k \leq \left[\frac{2(n-b)}{d+3} \right], \quad k \leq \left[\frac{2b}{d+1} \right], \quad f(n, b, d) \leq bd + k \quad \blacksquare \quad 7$$

yield the desired results (8), (9).

LEMMA 3.

$$f(n, b, d) \leq \max\left\{\binom{2b+1}{2}, \left\lfloor \frac{b(d+n-b)}{2} \right\rfloor\right\}.$$

Proof: Let n, b, d be given and let (10) be a partition which maximizes (11). Then Lemma 1 yields

$$f(n, b, d) \leq \frac{n_0(d+n-n_0)}{2} + \sum_{i=1}^m \binom{n_i}{2} \leq \frac{n_0(d+n-n_0)}{2} + \binom{n-n_0-(m-1)}{2}.$$

This can be written as $f(n, b, d) \leq H(n_0)$ where

$$H(n_0) = \frac{n_0(d+n-n_0)}{2} + \binom{2b-2n_0+1}{2}$$

is a convex function of n_0 . Since $n_0 = n - \sum_{i=1}^m n_i \leq n-m = 2b-n_0$, we have $0 \leq n_0 < b$. Therefore

$$f(n, b, d) \leq \max\{H(0), H(b)\} = \max\left\{\binom{2b+1}{2}, \left\lfloor \frac{b(d+n-b)}{2} \right\rfloor\right\}$$

and the desired result follows by integrality of $f(n, b, d)$.

3: Constructions

LEMMA 4. If d is odd, $n > 2b$ and

$$\left\lceil \frac{2(n-b)}{d+3} \right\rceil (d-1) > (n-2b)d \quad (13)$$

then $f(n, b, d) \geq \left\lceil \frac{nd}{2} \right\rceil$.

Proof: Set $m = \lceil \frac{2(n-b)}{d+3} \rceil$ and $n_0 = 2b-n+m$. As $2b \leq n$, we have

$$2b(1 - \frac{1}{d+3}) \leq n(1 + \frac{1}{d+1} - \frac{2}{d+3})$$

and so

$$n_0 = 2b - n + \lceil \frac{2(n-b)}{d+3} \rceil \leq \frac{n}{d+1}$$

which can be written as

$$n_0^d < n - n_0 .$$

Besides,, (13) yields $n_0^d \geq m$. Now, let us set

$$a = \begin{cases} dn_0 & \text{if } n \text{ is even ,} \\ dn_0 - 1 & \text{if } n \text{ is odd .} \end{cases}$$

We have

$$a \equiv dn_0 + n \equiv n_0 + n \equiv n_0 + n - 2b \equiv m \pmod{2} \quad (14)$$

and

$$n - n_0 \geq dn_0 \geq a \geq dn_0 - 1 \geq m - 1 . \quad (15)$$

Set $n_i = d+2$ for $i = 1, 2, \dots, m-1$ and

$$n_m = n - \sum_{i=0}^{m-1} n_i = 2n - 2b - m(d+3) + d+2 > d+2 .$$

By (14) and (15), $a-m$ is an even nonnegative integer not exceeding

$$\sum_{i=1}^m (n_i - 1) . \quad \text{Let } s \text{ be the greatest integer with } a-m \geq \sum_{i=1}^s (n_i - 1) ;$$

then $0 < s \leq m$. Set

$$a_i = \begin{cases} n_i & \text{if } 1 \leq i \leq s, \\ a-m - \sum_{i=1}^s (n_i - 1) + 1 & \text{if } i = s+1, \\ 1 & \text{if } s+1 < i < m. \end{cases}$$

Obviously, each a_i is odd and $\sum_{i=1}^m a_i = a$. Take disjoint graphs G_1, G_2, \dots, G_m where each G_i has exactly n_i vertices, a_i of them of degree $d-1$ and the remaining $n_i - a_i$ of degree d . The a vertices of $G_1 \cup G_2 \cup \dots \cup G_m$ having degree $d-1$ will be enumerated as

u_1, u_2, \dots, u_a . Add a new set S of n_0 vertices v_1, v_2, \dots , join each v_i to all the vertices u_j with $(i-1)d < j \leq \min(id, a)$ and call the resulting graph G .

If $a = dn_0$ (i.e., if n is even) then all the n vertices of G have degree d ; if $a = dn_0 - 1$ (i.e., if n is odd) then $n-1$ vertices of G have degree d and the last one has degree $d-1$. In both cases, G has $\lceil \frac{a^2}{2} \rceil$ edges. Since $k_0(G-S) > m$, G contains at most b independent edges.

LEMMA 5. If d is odd, $n > 2b$ and

$$\lceil \frac{2(n-b)}{d+3} \rceil (d-1) < (n-2b)d \quad (16)$$

then

$$f(n, b, d) > bd + \lceil \frac{2(n-b)}{d+3} \rceil$$

Proof: Set $m = \lceil \frac{2(n-b)}{d+3} \rceil$ and $n_0 = 2b - n + m$. Then (16) yields $n_0 d < m$. Set $n_i = d+2$ for $i = 1, 2, \dots, m-1$ and

$$n_m = n - \sum_{i=0}^{m-1} n_i = 2n - 2b - m(d+3) + d + 2 > d + 2 .$$

Take disjoint graphs G_1, G_2, \dots, G_m where each G_i has n_i-1 vertices of degree d and one vertex u_i of degree $d-1$. Add a new set S of n_0 vertices v_1, v_2, \dots , join each v_i to all the vertices u_j with $(i-1)d < j \leq id$ and call the resulting graph G . Obviously, all but $m-n_0$ vertices of G have degree d ; the remaining $m-n_0$ vertices have degree $d-1$. Hence G has exactly

$$\frac{1}{2} (nd - (m-n_0)d) = bd + m \cdot \frac{d-1}{2}$$

edges. Since $k_0(G-S) = m = n - 2b + |S|$, G contains at most b independent edges.

To make this paper self-contained, we need three more lemmas; these are due to Hanson [6].

LEMMA 6. If $d \leq 2b$ and $n \geq 2b + \left\lceil \frac{b}{\lceil \frac{d+1}{2} \rceil} \right\rceil$ then

$$f(n, b, d) \geq bd \left\lceil \frac{b}{\lceil \frac{d+1}{2} \rceil} \right\rceil \cdot \left\lceil \frac{d}{2} \right\rceil .$$

Proof: Case 1, d odd. Set $m = \lceil \frac{2b}{d+1} \rceil$, $n_i = d+2$ for $i = 1, 2, \dots, m-1$ and

$$n_m = 2b + m - (m-1)(d+2) = 2b - m(d+1) + d + 2 > d + 2 .$$

Take disjoint graphs G_1, G_2, \dots, G_m where each G_i has n_i-1 vertices of degree d and one of degree $d-1$. Add $n-(2b+m)$ isolated vertices and call the resulting graph G . Clearly, G has

$$\frac{1}{2} ((2b+m)d-m) = bd + m \cdot \frac{d-1}{2}$$

edges and at most

$$\sum_{i=1}^m \left[\frac{n_i}{2} \right] = \sum_{i=1}^m \frac{n_i - 1}{2} = \frac{1}{2} \left(\sum_{i=1}^m n_i - m \right) = b$$

independent edges.

Case 2, d even. Set $m = [\frac{2b}{d}]$, $n_i = d+1$ for $i = 1, 2, \dots, m-1$ and

$$n_m = 2b + m - (m-1)(d+1) = 2b - md + d + 1 > d + 1 .$$

Take disjoint graphs G_1, G_2, \dots, G_m where each G_i has n_i vertices, all of degree d . Add $n - (2b + m)$ isolated vertices and call the resulting graph G . Clearly, G has

$$\frac{1}{2} (2b + m) \cdot d = bd + m \quad \text{§}$$

edges and at most

$$\sum_{i=1}^m \left[\frac{n_i}{2} \right] = \sum_{i=1}^m \frac{n_i - 1}{2} = \frac{1}{2} \left(\sum_{i=1}^m n_i - m \right) = b$$

independent edges.

LEMMA 7. If d is even, $d \leq 2b$ and $n \leq 2b + [\frac{2b}{d}]$ then $f(n, b, d) \geq \frac{nd}{2}$.

Proof: Set $m = n - 2b$; then $m(d+1) \leq n$. For each $i = 1, 2, \dots, m-1$, set $n_i = d+1$; set also $n_m = n - (m-1)(d+1) > d+1$. Let G be a disjoint union of graphs G_1, G_2, \dots, G_m where each G_i has n_i vertices, all of degree d . Then G has exactly $\frac{1}{2} dn$ edges and at most

$$\sum_{i=1}^m \left[\frac{n_i}{2} \right] = \sum_{i=1}^m \frac{n_i - 1}{2} = \frac{1}{2} (n - m) = b$$

independent edges.

LEMMA 8.

(i) If $d \geq 2b$, $n \geq 2b+1$ then $f(n, b, d) \geq \binom{2b+1}{2}$.

(ii) If $d > b$, $d+1 \leq n \leq d+b$ then $f(n, b, d) > \left[\frac{b(n+d-b)}{2} \right]$.

(iii) If $d > b$, $n \geq b+d$ then $f(n, b, d) \geq bd$.

Proof:

(i) Take a complete graph with $2b+1$ vertices, add $n-(2b+1)$ isolated vertices.

(ii) If $b(d-n+b)$ is odd, take a graph G_0 with $b-1$ vertices of degree $d-n+b$ and one of degree $d-n+b-1$. If $b(d-n+b)$ is even, take a graph G_0 with b vertices of degree $d-n+b$. Add $n-b$ new vertices, join each of them to all the vertices of G_0 and call the resulting graph G . Obviously, the degrees of vertices of G do not exceed $\max\{d, b\} = d$; since each edge of G has at least one endpoint in G_0 , we conclude that G has at most b independent edges. Finally, G has exactly

$$\left[\frac{b(d-n+b)}{2} \right] + b(n-b) = \left[\frac{b(n+d-b)}{2} \right]$$

edges.

(iii) Take a complete bipartite graph with b vertices in one part and d in the other; add $n-(b+d)$ isolated vertices.

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