EDMONDS POLYHEDRA AND WEAKLY HAMILTONIAN GRAPHS

BY

VACLAV CHVATAL

STAN-CS-72-256

JANUARY, 1972

COMPUTER SCIENCE DEPARTMENT

School of Humanities and Sciences

STANFORD UNIVERSITY

EDMONDS POLYHEDRA AND WEAKLY HAMILTONIAN GRAPHS

by V. Chvátal

Abstract: Jack Edmonds developed a new way of looking at extremal combinatorial problems and applied his technique with a great success to the problems of the maximal-weight degree-constrained subgraphs. Professor C. St. J. A. Nash-Williams suggested to use Edmonds' approach in the context of hamiltonian graphs. In the present paper, we determine a new set of inequalities (the "comb inequalities") which are satisfied by the characteristic functions of hamiltonian circuits but are not explicit in the straightforward integer programming formulation. A direct application of the linear programming duality theorem then leads to a new necessary condition for the existence of hamiltonian circuits; this condition appears to be stronger than the previously known ones. Relating linear programming to hamiltonian circuits, the present paper can also be seen as a continuation of the work of Dantzig, Fulkerson and Johnson on the travelling salesman problem.

AMS (MOS) subject classification numbers (1970): Primary 05C35, 90C10

Keywords and phrases: Hamiltonian circuits, integer linear programming, travelling salesman problem, matching polyhedron.

This research was supported in part by the National Science Foundation under grant number GJ 304084, and the Office of Naval Research under contract number N-00014-67-A-0112-0057 NR 044-402.

0. Introduction

During my work on this paper, I enjoyed useful and inspiring discussions with Professor J. A. Bondy, Professor Jack Edmonds, Dr. Steve Gallant, Professor C. St. J. A. Nash-Williams and Professor Richard Rado. In particular, Professor Nash-Williams suggested to explore the relations between hamiltonian circuits and linear programming and to use the term "weakly hamiltonian graphs" for graphs admitting certain functions related to hamiltonian circuits. I also thank Miss Laurel L. Ward of McGill University for computing assistance.

If V is a set, we define $[V] = \{A \subset V: |A| = 2\}$. A graph is an ordered pair $G = (V,X)$ where V is a set and $X \subset [V]$. All the graph-theoretical definitions not given here can be found in [12]. A graph is n-cyclable if given any set $S \subset V$ with $|S| = n$ there is a cycle passing through all points of S . A graph is t-tough if, for each set $S \subset V$, the S-deleted subgraph $G-S$ has at most $\max\{|S|/t, 1\}$ components (see [2]). If T,W are disjoint sets, we define $[T,W] = \{A: |A| = 2, A \cap T \neq \emptyset, A \cap W \neq \emptyset\}$. For a fixed graph $G = (V,X)$ and sets $T,W \subset V$, we set $q(T) = |X \cap [T]|$, $q(T,W) = |X \cap [T,W]|$. The subgraph $(T, X \cap [T])$ induced by T will be denoted by $G(T)$; the number of components of $G(T)$ will be denoted by $k(T)$.

If V is a set, we denote by $\exp^* V$ the set of all proper non-empty subsets of V ; we denote by $\exp_o V$ the set of all odd-cardinality subsets of V . We denote by N the set of all nonnegative integers. If f is a real-valued function defined on S then we write $f \cdot T$ rather than $\sum_{x \in S \cap T} f(x)$.

1. Edmonds polyhedra

Let us begin with a set of inequalities

$$\sum_{i=1}^{n} a(i,j)x(i) \leq b(j) \qquad (j = 1,2, \ldots ,m) \tag{1}$$

(a(i,j), b(j) being real numbers) which determine a bounded subset of the n-dimensional Euclidean space R^n . Then the set S of the lattice points of R^n (i.e., the points

$$x = (x(1),x(2),\ldots,x(n))$$

where the x(i)'s are integers) satisfying (1) is finite. Its convex hull is a polyhedron which can be characterized by a new set of inequalities

$$\sum_{i=1}^{n} a^*(i,j)x(i) \leq b^*(j) \qquad (j = 1,2,\ldots,m^*) . \tag{2}$$

The polyhedron determined by (1) will be denoted by P , the polyhedron determined by (2) will be denoted by E(P) .

Next, consider the following couple of linear programming problems:

$$\text{- maximize } \sum_{i=1}^{n} c(i)x(i) \text{ subject to } x \in S \tag{3}$$

$$\text{- maximize } \sum_{i=1}^{n} c(i)x(i) \text{ subject to } x \in E(P) \tag{4}$$

Since the vertices of E(P) come from S and S is a subset of E(P) , we have

$$\max_{x \in S} \sum c(i)x(i) = \max_{x \in E(P)} \sum c(i)x(i) .$$

Therefore every optimal solution of (3) is an optimal solution of (4). Conversely, every <u>basic</u> optimal solution of (4) is an optimal solution of (3).

Hence, if we know how to pass from the inequalities (1) to the inequalities (2), we can reduce the integer linear programming problem (3) to the ordinary (continuous) linear programming problem (4). Naturally, starting from the inequalities (1), one can always determine the (finite) set S and, in turn, the inequalities (2). In practice, however, this process may be extremely lengthy.

Apparently the only case where (2) has been explicitly determined for quite a wide class of polyhedra (1) is the case of the maximum-weight degree-constrained subgraphs. Here one begins with a graph $G = (V,X)$ and a weight-function $c: E \to R$. The problem of finding a maximum-weigh-t matching in G is the following integer linear programming problem: maximize $\sum_{x \in X} c(x)f(x)$ subject to

$$f\cdot[\{u\},V] \leq 1 \quad (u \in V) \tag{5}$$

$$f(x) \geq 0 \quad (x \in X) \tag{6}$$

$$f(x) = \text{integer} .$$

The inequalities (5), (6) determine a polyhedron P in the Euclidean space R^X . Edmonds [6] proved that E(P) is characterized by the inequalities (5),(6), and

$$f\cdot[W] \leq \frac{1}{2}(|W|-1) \quad (W \in \exp_0 V) . \tag{7}$$

Recently, another proof of this theorem has been found by Balinski [1]. The inequalities (5) generalize into

$$f\cdot[\{u\},V] \leq d(u) \qquad (WV) \tag{8}$$

where d is an arbitrary function $d: V \to N$. Every integer-valued function f satisfying (8) and the inequalities

$$0 < f(x) \leq 1 \qquad (x\epsilon X) \tag{9}$$

satisfies necessarily the inequalities

$$f\cdot([W]\cup Y) \leq \left[\frac{d\cdot W + |Y|}{2}\right] \qquad (W \epsilon \exp V, Y \subset [W, V-W]) \ . \tag{10}$$

Indeed, (8) and (9) imply

$$f\cdot([W]\cup Y) \leq \frac{1}{2}\left(\sum_{u\epsilon W} f\cdot[\{u\},V] + f\cdot Y\right) \leq \frac{1}{2}(d\cdot W + |Y|)$$

and (10) follows by the integrality of f. (Let us note that (7) is a special case of (10) with $d \equiv 1$, $Y = \emptyset$ and $|W|$ odd.)

Conversely, Jack Edmonds proved ([6], Section 8, polyhedron II) that, if P is defined by (8) and (9), then E(P) is determined by (8), (9) and (10) . Now, the duality theorem of linear programming implies that

$$\max f\cdot X = \min z$$

where f ranges over all <u>integer-valued</u> functions satisfying (8), (9) and

$$z = \sum_{u\epsilon V} a(u)d(u) + b\cdot X + \sum_{D} c(W,Y)\left[\frac{d\cdot W + |Y|}{2}\right]$$

where a,b,c ranges over all functions

$$\left.\begin{array}{l} a: V \to [0,\infty) \\ b: X \to [0,\infty) \\ c: D = \{(W,Y): W \in \exp V, Y \subset [W,V-W]\} \to [0,\infty) \end{array}\right\} \qquad (11)$$

subject to the constraints

$$\sum_{u \in X} a(u) + b(x) + \sum_{x \in [W] \cup Y} c(W,Y) \geq 1 \quad (x \in X) \quad . \qquad (12)$$

Actually, one can choose the functions (11) minimizing z in a very particular manner (see [8], Theorem (19)). Indeed, it can be shown that

$$\max f \cdot X = \min(d \cdot S + q(T) + \sum_{W} [\frac{d \cdot W + q(W,T)}{2}]) \qquad (13)$$

where the minimum ranges over all partitions $V = R \cup S \cup T$ and the summation is extended over all (point-sets of) components of $G(R)$. Now, with each partition $V = R \cup S \cup T$, one can associate the functions (11) by setting

$$a(u) = \begin{cases} 1 & \text{if } u \in S \\ 0 & \text{otherwise} \end{cases}$$

$$b(x) = \begin{cases} 1 & \text{if } x \in [T] \\ 0 & \text{otherwise} \end{cases}$$

$$c(W,Y) = \begin{cases} 1 & \text{if } W \text{ is a component of } G(R) \text{ and } Y = [W,T] \\ 0 & \text{otherwise.} \end{cases}$$

Then the constraints (12) are satisfied and z becomes the expression in the right-hand side of (13). In a similar fashion, Berge's maximal matching formula [2] (see also [7], Section 5.6) relates to the polyhedron (5), (6), (7).

At this point, a proper credit should be given to Professor W. T. Tutte. Berge's maximal matching formula and the formula (13) appear to be just corollaries to his factor theorems contained in [13] and [14].

Edmonds made it quite explicit (see Section 5 of [7] or Section 1 of [6]) that his approach to maximal-weight degree-constrained subgraphs can be applied in other settings. Indeed, many combinatorial problems -- from the four-color conjecture to the determination of Ramsey numbers to the questions of existence of block-designs -- are essentially integer linear programming problems. In each case, the polyhedron P is defined in a rather simple way. The above examples show how the knowledge of the corresponding polyhedron E(P) -- which will be called the Edmonds polyhedron of P -- combined with the duality theorem of linear programming could be used in solving each of these problems. Professor Richard Rado pointed out to me that even the knowledge of an in-between polyhedron contained in P and containing E(P) -- or equivalently, the knowledge of inequalities which are implied by (2) but not by (1) alone -- could serve as a heuristic tool in obtaining correct combinatorial results. (In the next section, we determine such an in-between polyhedron related to hamiltonian circuits in graphs.) In this context, we mention recent work of Hammer [11] who uses Boolean functions to characterize the lattice points contained in P .

There is also a link between the Edmonds' polyhedra approach and Gomory's integer linear programming algorithm [9], [10]. (Gomory's "cutting planes" correspond to the added inequalities in (2).) However, a more detailed discussion of this link exceeds the scope of the present paper.

2. Weakly hamiltonian graphs

Let $G = (V,X)$ be a graph. Obviously, the characteristic function $f: x \to \{0,1\}$ of a hamiltonian circuit in G (if there is any) satisfies the contraints

$$0 \leq f(x) \quad (x \in X) \tag{14}$$

$$\sum_{u \in x} f(x) \leq 2 \quad (u \in V) \tag{15}$$

$$f \cdot [Q] \leq |Q| - 1 \qquad (Q \in \exp^* V) \quad . \tag{16}$$

By a comb in G we will mean a set

$$K = \bigcup_{i=0}^{n} [W_i] \tag{17}$$

where W_i's are subsets of V with $W_i \neq V$ and $|W_i \cap W_0| = 1$ for all $i = 1,2,\ldots,n$. The inequalities (15), (16) and the integrality of f imply that

$$f \cdot \bigcup [W_i] \leq |W_0| + \sum_{i=1}^{n} (|W_i| - 1) - \{\tfrac{n}{2}\} \tag{18}$$

for each comb (17) in G . Indeed, one has

$$2f \cdot \bigcup \ [Wi] \leq \sum_{u \in W_0} \sum_{u \in x} f(x) + \sum_{i=1}^{n} f \cdot [W_i] + \sum_{i=1}^{n} f \cdot [W_i - W_0] \leq$$

$$\leq 2|W_0| + \sum_{i=1}^{n} (|W_i| - 1) + \sum_{i=1}^{n} (|W_i| - 2) =$$

$$= 2|W_0| + 2 \sum_{i=1}^{n} (|W_i| - 1) - n$$

so that

$$f\cdot \cup [W_i] \le |W_0| + \sum_{i=1}^{n} (|W_i|-1) - \frac{n}{2} \quad . \tag{19}$$

Now (18) follows since the left-hand side of (19) is an integer. The right-hand side of (18) will be called the <u>rank</u> of the comb (17) and denoted by $r(K)$; then (18) can be written as

$$f\cdot K \le r(K) \quad . \tag{18'}$$

In particular, if each W_i $(i = 1,2,\ldots,n)$ contains just two vertices and these vertices are adjacent then

$$Y = \bigcup_{i=1}^{n} [W_i] \subset [W_0, V-W_0]$$

and (18) reduces into

$$f\cdot([W_0] \cup Y) \le |W_0| + [\tfrac{n}{2}] = |W_0| + [\tfrac{1}{2}|Y|].$$

The last inequality is also a special case of (10) with $d \equiv 2$.

By a <u>weakly hamiltonian function</u> on G we will mean any function $f\colon X \to [0,\infty)$ which satisfies (14), (15), (16), (18). G will be called <u>weakly hamiltonian</u> if it admits a weakly hamiltonian function f with $f\cdot X = |V|$. As we have shown, the characteristic function of a hamiltonian circuit is weakly hamiltonian and so every hamiltonian graph is also weakly hamiltonian. The duality theorem of linear programming yields at once the following characterization of weakly hamiltonian graphs.

A graph $G = (V,X)$ is not weakly hamiltonian if and only if there are functions

$$a: V \to [0,\infty)$$

$$b: \exp^* V \to [0,\infty) \qquad (20)$$

$$c: D \to [0,\infty)$$

(where D is the set of all combs in G) such that

$$\sum_{u\in x} a(u) + \sum_{x\in[Q]} b(Q) + \sum_{x\in K} c(K) \geq 1 \quad (x\in X) \qquad (21)$$

and $z < |V|$ where

$$z = 2\sum_V a(u) + \sum_{\exp^* V} (|Q|-1)b(Q) + \sum_D r(K)c(K) \ .$$

Restricting ourselves to a rather special subclass of functions (20) we obtain a weaker theorem which may, however, seem to be more elegant.

<u>THEOREM 1</u>. If $G = (V,X)$ is weakly hamiltonian then there is no partition $V = R \cup S \cup T$ into pair-wise disjoint (possibly empty) sets with $T \neq V$ and

$$|S| + \sum_H [\tfrac{1}{2} q(H,T)] < k(T) \qquad (22)$$

where the summation is extended over all ccxnponents H of G(R) .

<u>Proof</u>: Assume that there is a partition $V = R \cup S \cup T$ with $T \neq V$ which satisfies (22). Define the functions (20) by

$$a(u) = \begin{cases} 1 & \text{if } u \in S \\ 0 & \text{otherwise} \end{cases}$$

$$b(Q) = \begin{cases} 1 & \text{if } (Q,[Q]) \text{ is a component of } G(T) \\ 0 & \text{otherwise} \end{cases}$$

$$c(K) = \begin{cases} 1 & \text{if } (W_0, [W_0]) \text{ is a component of } G(R) \\ & \text{and } \{W_i : 1 \le i \le n\} = [W_0,T] \\ 0 & \text{otherwise} \end{cases}$$

Then the constraints (10) are satisfied and we have

$$z = 2|S| + (|T| - k(T)) + \sum_K [W_0 + \tfrac{1}{2} q(W_0,T)] =$$

$$= 2|S| + |T| - k(T) + |R| + \sum_H [\tfrac{1}{2} q(H,T)] =$$

$$= |V| + |S| + \quad [\tfrac{1}{2} q(R,T)] - k(T) < |V|$$

so that G is not weakly **hamiltonian.**

<u>THEOREM 2.</u> Let $G = (V,X)$ be a graph and m a positive integer. Let there be subsets $W, W_0, W_1, \ldots, W_{2m+1}$ of V such that

$$V = W \cup \left(\bigcup_{i=0}^{2m+1} W_i \right) , \quad X = [W,V] \cup \left(\bigcup_{i=0}^{2m+1} [W_i] \right)$$

$$|W| = m \ , \quad W_i \cap W = \emptyset \qquad (i = 0,1,\ldots,m)$$

$$i > 0 \Rightarrow |W_i \cap W_0| = 1 \ , \quad |W_i| \ge 2$$

$$i > j > 0 \Rightarrow W_i \cap W_j = \emptyset \ .$$

Then G is not weakly hamiltonian.

Proof: Define the functions (20) by

$$a(u) = \begin{cases} 1 & \text{if } u \in W \\ 0 & \text{otherwise} \end{cases}$$

$$b(Q) = 0 \quad \text{for all } Q \in \exp^* V$$

$$c(K) = \begin{cases} 1 & \text{if } K = \bigcup_{i=0}^{2m+1} [W_i] \\ 0 & \text{otherwise} \end{cases}$$

Then the constraints (21) are satisfied and we have

$$z = 2|W| + |W_0| + \sum_{i=1}^{2m+1} (|W_i|-1) - \{\frac{2m+1}{2}\} = |V|-1$$

so that G is not weakly hamiltonian.

THEOREM 3. Let G be a graph and n a positive integer such that

$$V = \bigcup_{i=1}^{2n+1} W_i \quad , \quad x = \bigcup_{i=1}^{2n+1} [W_i]$$

$$W_1 \cap W_2 = \emptyset \quad , \quad 3 \leq i < j \Rightarrow W_i \cap W_j = \emptyset$$

$$i < 2 < j \Rightarrow |W_i \cap W_j| = 1 .$$

Then G is not weakly **hamiltonian.**

Proof: Define the functions (20) by

$$a(u) = \begin{cases} \frac{1}{3} & \text{if } u \in W_1 \cup W_2 \\ 0 & \text{otherwise} \end{cases}$$

$$b(Q) = \begin{cases} \frac{1}{3} & \text{if } Q = W_i - (W_1 \cup W_2)\ , \quad i \geq 3 \\ 0 & \text{otherwise} \end{cases}$$

$$c(K) = \begin{cases} \frac{1}{3} & \text{if } K = [W_i] \cup \left(\bigcup_{j=3}^{2n+1} [W_j] \right)\ , \quad i \in \{1,2\} \\ 0 & \text{otherwise .} \end{cases}$$

Then the constraints (21) are satisfied and we have

$$z = \frac{2}{3}(|W_1| + |W_2|) + \frac{1}{3}\sum_{i=3}^{2n+1}(|W_i|-3) + \frac{1}{3}\sum_{i=1}^{2}(|W_i| + \sum_{j=3}^{2n+1}(|W_j|-1) - n)$$

$$= \sum_{i=1}^{2n+1}|W_i| - (4n - \frac{5}{3}) = |V| - \frac{1}{3}$$

so that G is not weakly hamiltonian.

COROLLARY (One-two-three theorem). If G is weakly hamiltonian then

(i) G is 1-tough,

(ii) G has a 2-factor,

(iii) G is 3-cyclable.

Proof: (i) If G is not 1-tough then there is a set $S \subset V$ with $k(G-S) > \max\{|S|/t, 1\}$. Evidently, S can be chosen to be non-empty. Setting $T = V-S$ and $R = \emptyset$ we obtain a partition as in Theorem 1 which satisfies (22). Therefore G is not weakly hamiltonian.

(ii) Every weakly hamiltonian function f satisfies the constraints (8), (9) and (10) with $d \equiv 2$. From Edmonds' theorem discussed in Section 1, it follows immediately that every weakly hamiltonian graph has a 2-factor.

An alternative proof makes use of Theorem 1 and **Tutte's** factor theorem [14]. Let $G = (V,X)$ be a weakly hamiltonian graph with no 2-factor. Then, by **Tutte's** theorem, G admits a partition $V = R \cup S \cup T$ with

$$|S| + \sum \left[\tfrac{1}{2} q(H,T) \right] < |T| - q(T) \quad . \tag{23}$$

Since $|T| - q(T) \leq k(T)$, Theorem 1 implies $T = V$. But then (23) reads $|X| = q(T) < |T| = |V|$. Hence G cannot be 1-tough (not even **2-connected**) contradicting (i).

(iii) Watkins and Mesner [15] characterized graphs which are not **3-cyclable.** Their theorem can be stated as follows: If G is not **3-cyclable** then either

(A) G is not 2-connected or

(B) there is a set $S \subset V$ with $|S| = 2$, $k(G-S) \geq 3$ or

(C) G is a graph of Theorem 2 with $m = 1$ or

(D) G is a graph of Theorem 3 with $n = 2$.

In the first two cases, G is not 1-tough and so, by (i), not weakly **hamiltonian.** In case (C), G is not weakly hamiltonian by Theorem 2, in case (D), G is not weakly **hamiltonian** by Theorem 3.

3. Afterthoughts

(1) Professor Jack Edmonds drew my attention to an interesting observation which is closely related to his concept of a good characterization (as explained in [5]). A good characterization of graphs having a 1-factor is provided by Tutte's theorem [13]. Once a 1-factor in (V,X) is exhibited, it is easy to check that it is a 1-factor indeed. On the other hand, if (V,X) has no 1-factor then there is a set $S \subset V$ such that the number $k_0(S)$ of odd components (i.e., components having odd number of points) of $G(V-S)$ exceeds $|S|$. Again, once such a set S is exhibited, it is easy to compute $k_0(S)$ and check the inequality $k_0(S) > |S|$.

With hamiltonian and weakly hamiltonian graphs, the situation is different; besides, these two cases are -- in a way -- complementary to each other. It is easy to recognize a hamiltonian circuit in a given graph (although it may be exceedingly difficult to find one) but so far we know no good way of recognizing that there is no such circuit. On the other hand, it may be exceedingly difficult to check that a given graph is weakly hamiltonian -- indeed, the number of constraints put upon weakly hamiltonian functions grows very fast with the size of G. However, it is much easier to check that G is not weakly hamiltonian. Indeed, if G is not weakly hamiltonian then there exist functions (20) satisfying (21) and $z < |V|$; moreover, these functions can be chosen to have altogether at most $|X|$ nonzero values. To check (21) and $z < |V|$ is relatively easy.

(2) One can think of real-valued functions defined on X as of the points of the $|X|$-dimensional Euclidean space R^X. In this space, the hamiltonian functions are the lattice points in the polyhedron (14), (15), (16). The weakly hamiltonian functions form a polyhedron which is contained in (14), (15), (16) and contains the Edmonds polyhedron of (14), (15), (16). Finding other linear inequalities which are satisfied by all hamiltonian functions, one would arrive at a better definition of weakly hamiltonian graphs (so that the weakly hamiltonian functions in the new sense would constitute a proper subset of the weakly hamiltonian functions as defined here). This process could eventually lead to the determination of Edmonds polyhedra corresponding to (14), (15), (16). For instance, the Petersen graph (see Figure 1) is weakly hamiltonian (the corresponding weakly hamiltonian function is obtained by setting $f(x) \equiv \frac{2}{3}$) but not hamiltonian. (Using the "1-2-3 theorem", one can show that every weakly hamiltonian graph with less than ten points is hamiltonian.)

(Figure 1)

Hence one may try to find new linear inequalities, satisfied by all hamiltonian functions and violated by every function f that is defined on the line-set X of the Petersen graph and which satisfies $f \cdot X = 10$. However, here comes a bit of a surprise.

The 15 x 15 matrix

$$
\left\|\begin{matrix}
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & -\,1 & 0 & 1 & 1 & 0 & 0
\end{matrix}\right\|
$$

is nonsingular and its rows are the incidence vectors of hamiltonian paths in G (with lines enumerated as in Figure 1). Since the rows are linearly independent and satisfy the linear equation $\sum_{j=1}^{15} a_{ij} = 9$, the hyperplane $\sum_{j=1}^{15} x_j = 9$ contains one of the faces of E(P). Equivalently, the inequality

$$f \cdot X \leq 9 \tag{24}$$

<u>must</u> be included in the minimal set of inequalities describing E(P).

What happens here? We want to find a set of linear inequalities which would enable us to show -- via the easy part of **LP** duality theorem -- that the Petersen graph is **nonhamiltonian.** However, we find that one of the inequalities in this <u>complete</u> set is the inequality (24), which is equivalent to the desired conclusion. With a refined taste for pathetic exclamations, one can say that the vicious circle is closed. In order to prove that the Petersen graph is nonhamiltonian, we must assume that the Petersen graph is nonhamiltonian.

But is the situation really that bad? Let us have a look at another example. Let us consider the graph $G = (V,X)$ in Figure 2 and the 15×15 matrix

$$\begin{Vmatrix}
1&0&1&0&1&1&0&1&1&0&0&1&1&0&0\\
0&1&1&0&1&1&0&1&1&1&0&0&0&1&0\\
0&1&0&1&1&0&0&1&1&1&1&0&1&0&0\\
0&0&1&1&0&1&0&1&1&1&0&0&1&1&0\\
0&1&1&0&1&1&0&1&0&1&0&1&1&0&0\\
0&1&1&0&1&1&1&1&0&1&0&0&0&1&0\\
0&1&1&0&1&1&1&0&1&0&1&0&1&0&0\\
0&1&1&0&1&1&0&0&1&1&0&0&1&0&1\\
0&1&1&0&1&1&0&1&1&0&1&0&1&0&0\\
0&1&1&0&1&1&0&1&1&0&0&1&1&0&0\\
0&1&1&0&1&1&0&1&1&0&1&0&0&0&1\\
0&1&1&0&1&1&0&1&1&0&0&1&0&1&0\\
1&0&1&1&1&0&0&1&1&0&0&1&1&0&0\\
1&1&0&1&0&1&0&1&1&1&0&0&0&0&1\\
0&0&1&0&1&0&0&1&1&0&1&1&1&1&0,
\end{Vmatrix}$$

which is nonsingular. Again, the rows of this matrix are the incidence vectors of hamiltonian paths in G and so the inequality $f \cdot X \leq 8$ must be included in the description of the related Edmonds polyhedron. But turn the pages back to Theorem 3 before giving out more cries of despair.

(Figure 2)

This graph is not even weakly hamiltonian. To prove that G is nonhamiltonian, we only need the basic inequalities (15), (16), (18). Therefore the mere presence of the inequality

$$f \cdot X \leq |V| - 1 \tag{25}$$

among those determining the Edmonds polyhedron of hamiltonian functions on G is not as disastrous as it may have seemed to be.

Nevertheless, our observations seem to indicate that the chances of determining completely the Edmonds polyhedra associated with hamiltonian functions may be quite low. There may be a jungle of graphs which are weakly hamiltonian yet not hamiltonian and which require the inequality (25) to be included in the complete description of the related polyhedra. (The Petersen graph is one of them, but who knows, there may be much worse ones.) If this is the case then the <u>complete</u> description of the polyhedra would necessarily involve showing that the jungle consists of nonhamiltonian graphs only. And this by-product itself may be dangerously close to a characterization of nonhamiltonian graphs.

(3) If, contrary to all our pessimism, the Edmonds polyhedra of (14), (15), (16) were known then the travelling salesman problem would be reduced to a continuous linear programming problem. This approach to the travelling salesman problem has been adopted by Dantzig, Fulkerson, and Johnson [4]. They noticed that, for practical purposes, one can often manage just with the inequalities (15) and (16) in order to prove the minimality of a tour (i.e., a hamiltonian circuit in a line-weighted graph). In solving the 42-city problem, however, they were forced to use two more linear inequalities (pointed to them by I. Glicksberg of RAND Corp.). The graph (V,X) they dealt with was a complete graph with points $1,2,\dots,42$; the first of the two additional inequalities read

$$f\cdot X = 42 \quad 3 \; f\cdot(14,15) + f\cdot Y - f\cdot[S,V] \leq 0 \qquad (26)$$

where $S = \{15,16,19\}$ and Y is the set of lines

$$\{14,15\} \;,\; \{15,18\} \,,\; \{17,16\},\{16,18\},\{20,19\},\{19,18\} \,.$$

Actually, (26) is satisfied by every weakly hamiltonian function. To see this, set $W_0 = V - \{15,16,19,18\}$, $W_1 = \{14,15\}$, $W_2 = \{17,16\}$, $W_3 = \{20,19\}$. Then (18) implies

$$f\cdot U[W_i] \leq 38 + 3 - \{\tfrac{3}{2}\} = 39$$

and so

$$f\cdot X + f\cdot Y - f\cdot[S,V] = f\cdot(U[W_i] \cup [\{18\},V]) \leq 39 + 2 = 41 \,.$$

Therefore

$$f\cdot X + f(14,15) + f\cdot Y - f\text{-}\&\,v\,] \leq 42$$

and (26) follows.

As for the other condition -- denoted by 67 in [5] -- the situation is much more messy. I don't see any way of deducing this (more complicated) one from (15), (16) and (18); perhaps it is independent of them. If this is the case, a more general formulation of that condition would yield an improved definition of weakly hamiltonian functions. (Dantzig, Fulkerson and Johnson exhibit a non-integral function ([4], Figure 18) which satisfies (14), (15), (16) but violates their condition 67. Perhaps this function is weakly hamiltonian.)

(4) This paper should be considered as a work in progress. The idea, and the definition, of weakly hamiltonian graphs, is a dynamic one. It is the author's hope that more people will find more restrictive linear constraints on hamiltonian functions, improving thus the present definition of weakly hamiltonian functions and graphs. And one day this process may lead to -- well, let us not be over-ambitious.

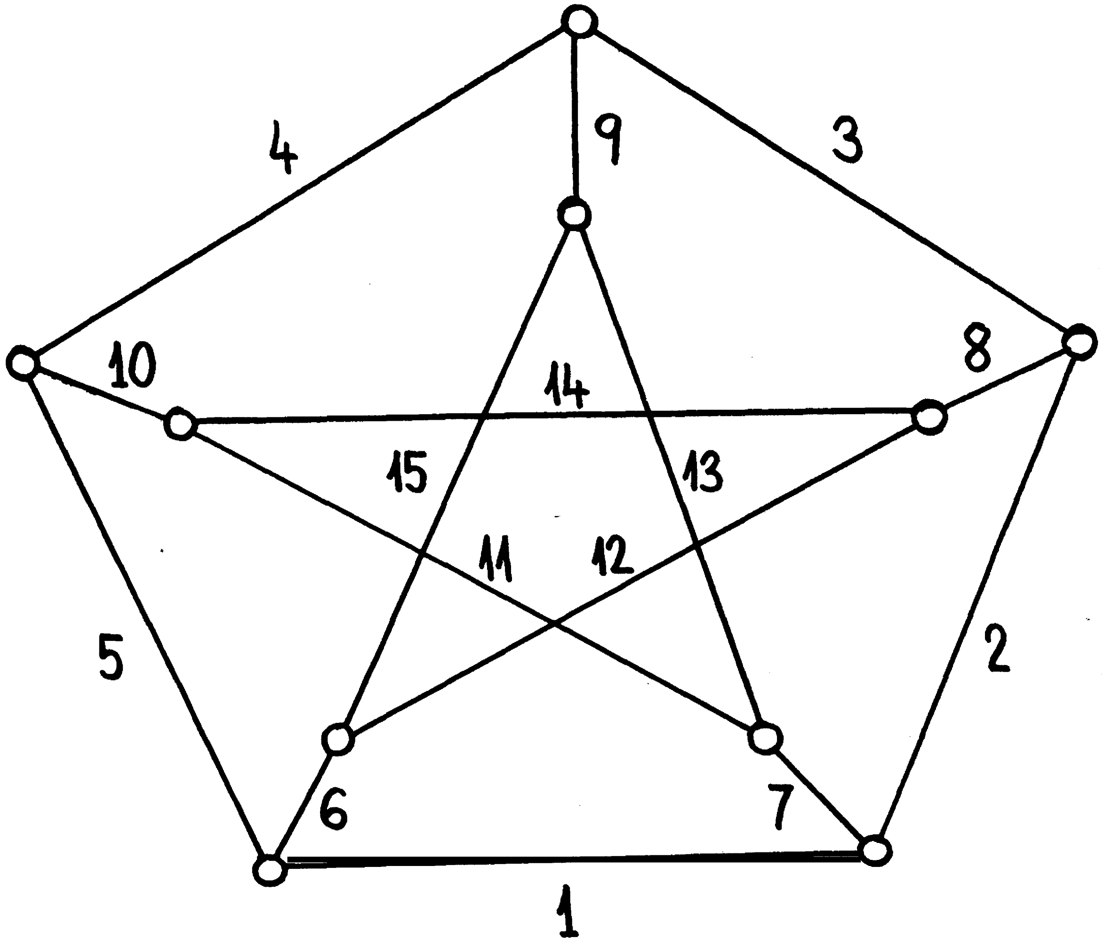

Fig. 1

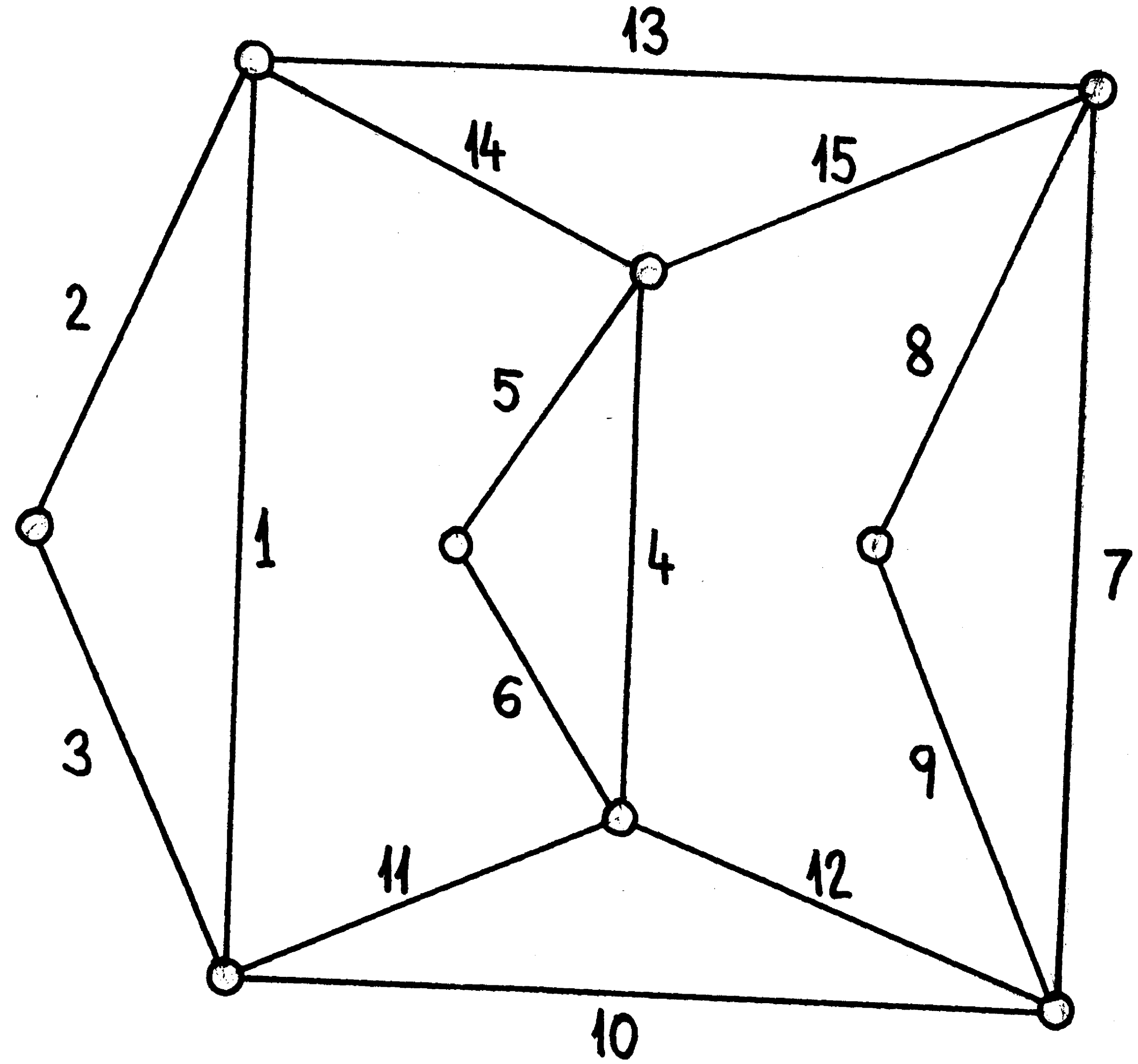

Fig. 2

References

[1] M. L. BALINSKI, Establishing the matching polyhedron, to appear in J. Combinatorial Theory.

[2] C. BERGE, Sur le couplage maximum d'un graphe, C. R. Acad. Sci., Paris 247 (1958), 258-259.

[3] V. CHVÁTAL, Tough graphs and hamiltonian circuits, to appear.

[4] G. DANTZIG, R. FULKERSON and S. JOHNSON, Solution of a large-scale traveling salesman problem, Op. Research 2 (1954), 393-410.

[5] J. EDMONDS, Minimum partition of a matroid into independent subsets, J. Res. Nat. Bur. Standards 69B (1965), 67-72.

[6] J. EDMONDS , Maximum matching and a polyhedron with 0,1-vertices, J. Res. Nat. Bur. Standards 69B (1965), 125-130.

[7] J. EDMONDS, Paths, trees, and flowers, Canad. J. Math. 17 (1965), 449-467.

[8] J. EDMONDS and E. L. JOHNSON, Matching: A well-solved class of integer linear programs, Combinatorial Structures and Their Applications, (R. K. Guy et al., Eds.), Gordon and Breach, New York 1970.

[9] R. E. GOMORY, Outline of an algorithm for integer solutions to linear programs, Bull. Amer. Math. Soc. 64 (1958), 275-278.

[10] R. E. GOMORY, An all-integer integer programming algorithm, Industrial Scheduling, 193-206, Prentice-Hall, Englewood Cliffs, N. J., 1963.

[11] P. HAMMER, BABO, A boolean approach for bivalent optimization, to appear.

[12] F. HARARY, Graph Theory, Addison-Wesley, Reading, Mass., 1969.

[13] W. T. TUTTE, The factorization of linear graphs, J. London Math. Soc. 22 (1947), 107-111.

[14] W. T. TUTTE, The factors of graphs, Canad. J. Math. 4 (1952), 314-328.

[15] M. E. WATKINS and D. M. MESNER, Cycles and connectivity in graphs, Canad. J. Math. 19 (1967), 1319-1328.