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Multiconic Surfaces

Abstract: Multiconic surfaces are a generalization of the type of surface called polyconic in numerical control of machine tools. The general theory is developed in this paper using a new parametrization. In the original form there was a problem as to whether or not a point that satisfies surface equations actually belonged to the intended surface. This difficulty is removed by the new technique.

Algorithms for calculation of line and plane intersections with the surface and for calculation of normal vectors, volume, and surface area are given for classes of defining functions of which it is required only that they have appropriate conditions of continuity and differentiability.

Examples are given of surfaces developed using spline functions. Preliminary comparative estimates of design and numerical control processing times are included.

Introduction

The APT language [1] developed for application to the control of machine tools includes a type of surface called "polyconic." There appears to be no adequate literature on the definition of such surfaces and their properties. The purpose of this paper is to provide such an exposition. The term "multiconic" is meant to imply that the control curves used are more general than the pseudopolynomials which define polyconic surfaces. This does not complicate the definition nor the development of certain properties of the surface.

Multiconic surfaces are finite connected surfaces for which a family of parallel planes intersect the surface in curves that are conic sections. For definiteness, let x, y, z be a rectangular coordinate system for which the set of parallel planes is perpendicular to the x axis with the domain of x finite (x_0, x_1). The mathematical definition of the surface then can be based on various ways of defining conic sections. Current practice includes at least four ways of doing this, all being related to a given triangle, as illustrated in Fig. 1.

All four of these ways assume that the conic passes through the points Q_0, Q_2 and is respectively tangent to the lines Q_0Q_1 and Q_2Q_1. From this information, a single conic can be specified by providing the eccentricity. Because eccentricity is inconvenient for present purposes another number, ϕ, is provided and is called "shape factor," to be defined later. Alternatively another point Q_3 on the conic, interior to the triangle, will complete the definition. This point is called the "shoulder."

Since by definition a multiconic is a set of such conics in parallel planes (perpendicular to the x axis), it follows that the set of points $\{Q_i\}$ for fixed i is a space curve. Each point of the set corresponds to a value of x, i.e., Q_i is a function of x, $Q_i(x)$. It is the practice, in defining such surfaces, to define such space curves in terms of their y and z coordinates, i.e.,

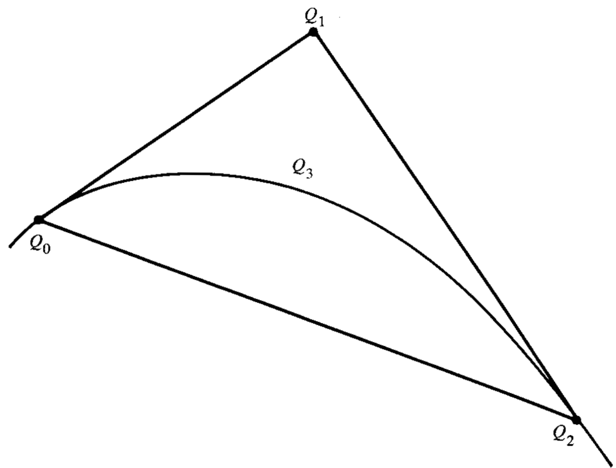
$$Q_i(x) \Longleftrightarrow (y_i(x), z_i(x)).$$

The functions $y_i(x)$ and $z_i(x)$ are called control curves. Likewise, the shape factor ϕ is $\phi(x)$. For notational convenience, the argument x is dropped whenever x is held constant.

Alternatively, the curve $Q_1(x)$ may be implicitly defined by a pair of scalar functions $S_0(x)$, $S_2(x)$ which represent functions of the angles at $Q_0(x)$, $Q_2(x)$.

Control curves of the form b $\sqrt{x} + \Sigma_{i=1}^{N} a_i x^i$, where N is not greater than eight, define polyconics. The reason for the square root term is to provide an infinite slope as,

Figure 1 Basic triangle defining a conic section.



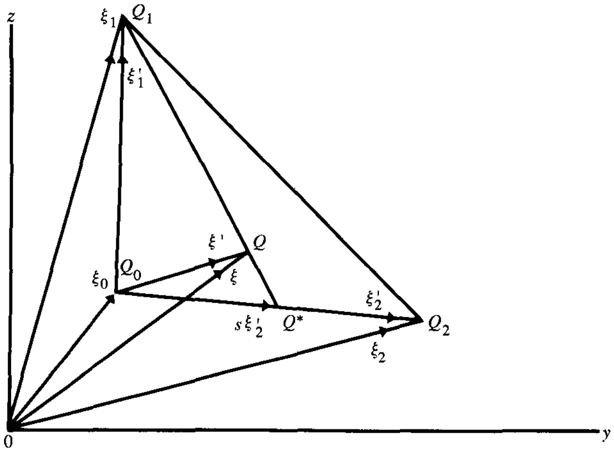


Figure 2 Vector construction of a conic section $Q(y, z)$.

for example, in the design of nose cones. Various users of the polyconic have added functions such as $ax + b \pm \sqrt{px^2 + qx + r}$ [2].

Such functions, however, are difficult to use for design and also are uneconomical for computing. Spline control curves [3, 4] greatly simplify the design process and also reduce computing costs.

In the remainder of this paper, line and plane intersections with the surface and the calculation of normal vectors, volume, and area are discussed and algorithms for their calculation provided. Line intersection and normal vectors are essential for numerical control data processing; plane intersections, volume, and area are essential for the process of designing surfaces.

It is assumed here that the various functions have any necessary continuity and differentiability.

Multiconic surface equations

For a fixed value of x, consider in Fig. 2 the points $Q(y, z)$, $Q_i(y_i, z_i)$, $i = 0, 1, 2$ determined by the vectors $\xi = (y, z)$, $\xi_i = (y_i, z_i)$. Let $\xi' = \xi - \xi_0$, $\xi_i' = \xi_i - \xi_0$, $y' = y - y_0$, $z' = z - z_0$, so that $\xi' = (y', z')$, $\xi_i' = (y_i', z_i')$.

Then ξ_1' is the vector from Q_0 to Q_1, ξ_2' from Q_0 to Q_2, and $s\,\xi_2'$ from Q_0 to Q^*. Evidently Q^* is on the segment Q_0Q_2 if and only if $0 \leq s \leq 1$. Otherwise it is on that segment extended, to the left if $s < 0$, and to the right if $s > 1$.

The vector $\xi_1' - s\,\xi_2'$ is the vector from Q^* to Q_1, and $t(\xi_1' - s\,\xi_2')$ is on that segment if and only if $0 \leq t \leq 1$; otherwise it is on that segment extended, and it is below if $t < 0$, and above if $t > 1$. Then

$$\xi' = s\xi_2' + t(\xi_1' - s\xi_2') = \xi_2'\, s(1-t) + \xi_1' t, \qquad (1)$$

$$\xi = \xi_2 s(1-t) + \xi_1 t + \xi_0 (1-s)(1-t), \qquad (2)$$

and certain assertions can be made about various regions of the plane.

In Fig. 2 the line through Q_0 and Q_2 corresponds to $t = 0$. The point Q_1 corresponds to $t = 1$. The line through Q_0, Q_1 corresponds to $s = 0$. The line through Q_2, Q_1 corresponds to $s = 1$.

As for the interiors of domains I, II, III in Fig. 3:

In I: $0 < s < 1, 0 < t < 1$,

In II: $0 < s < 1, t < 0$,

In III: $0 < s < 1, t > 1$.

Consider the function $g(s, t) = t^2 - \phi s(1-s)(1-t)^2$ and the equation

$$g(s, t) = 0. \qquad (3)$$

Now, if $E = (1/2)(y_2' z_1' - y_1' z_2') \neq 0$ then Eq. (1) can be solved for s, t:

$$s = (z_1' y' - y_1' z') / (2E + z_2' y' - y_2' z'),$$
$$t = -(z_2' y' - y_2' z')/2E \qquad (4)$$

for any point y', z', for which $2E + z_2' y' - y_2' z' \neq 0$. This inequality has the geometrical significance that Q cannot be on the line through Q_1 parallel to Q_0Q_2. When these values are substituted into (3), the equation becomes

$$(y_2' z' - z_2' y')^2 - \phi(z_1' y' - y_1' z')$$
$$\times [2E + (z_2' - z_1') y' - (y_2' - y_1') z'] = 0, \qquad (5)$$

which, being of second degree in y and z, is a conic.

If this equation is solved for ϕ, the result is

$$\phi = \begin{vmatrix} y_2' & z_2' \\ y' & z' \end{vmatrix}^2 \div \begin{vmatrix} y' & z' \\ y_1' & z_1' \end{vmatrix} \begin{vmatrix} y' & z' & 1 \\ y_2' & z_2' & 1 \\ y_1' & z_1' & 1 \end{vmatrix} \qquad (6)$$

or, in terms of the untranslated coordinates,

$$\phi = \begin{vmatrix} y_0 & z_0 & 1 \\ y_2 & z_2 & 1 \\ y & z & 1 \end{vmatrix}^2 \div \begin{vmatrix} y_0 & z_0 & 1 \\ y & z & 1 \\ y_1 & z_1 & 1 \end{vmatrix} \begin{vmatrix} y & z & 1 \\ y_2 & z_2 & 1 \\ y_1 & z_1 & 1 \end{vmatrix}. \qquad (7)$$

Geometrically, this is the square of the area of triangle Q_0Q_2Q divided by the product of the areas Q_0QQ_1 and QQ_2Q_1. The signs of the areas are, of course, important. Assuming that the vertices of the triangles are traversed in row order, the signs of the determinants are positive for counterclockwise traversal and negative for clockwise traversal. It follows that ϕ is positive in the interior of regions I, II, III, is zero on the line through Q_0Q_2 (except at Q_0 and Q_2), is not finite on the lines through Q_0 and Q_1, Q_2 and Q_1, and is negative elsewhere. Obviously, given Q_0, Q_1, Q_2, Q, then ϕ can be computed if the exceptions are excluded. This calculation is required when the shoulder curve is given instead of ϕ.

From elementary theory of conics it can be shown that

1. the conic is a straight line if $\phi = 0$,
2. an ellipse if $0 < \phi < 4$,
3. a parabola if $\phi = 4$,
4. a hyperbola if $\phi > 4$ or $\phi < 0$.

However, for the application being considered here, there is no interest in hyperbolas exterior to the domains I, II, III. Therefore the restriction $\phi \geq 0$ is made. Note that this excludes Q for which $2E + z_2'y' - y_2'z' = 0$ except at $Q = Q_1$, in which case ϕ is not finite.

In the event that $E = 0$, Eq. (4) is no longer valid. Equation (3) then defines a degenerate conic.

Another shape function $\rho = \sqrt{\phi}/(2 + \sqrt{\phi})$ is also used. Its principal utility, from the design point of view, is that $t = \rho$ when $s = 1/2$. Hence if a point Q is on the conic and on the line from Q_1 to $\overline{Q}$, the bisector of the base Q_0Q_2, then ρ is the length of $\overline{Q}Q$ divided by the length of $\overline{Q}Q_1$. For $\rho = 1$ the conic is a degenerate hyperbola. It is the pair of rays through Q_0, Q_1 and Q_2, Q_1.

It is necessary [5] to impose further restrictions on the functions $\phi(x)$ and $\rho(x)$ of the form

$$0 < \phi_1 \leq \phi(x) < \phi_2,$$
$$0 < \rho_1 \leq \rho(x) \leq \rho_2 < 1, \tag{8}$$

where $\phi_1, \phi_2, \rho_1, \rho_2$ are independent of x. For a zero value, the tangents at Q_0 and Q_2 would no longer be directed toward Q_1, whereas for any $\phi > 0$, $\rho > 0$, they are directed toward Q_1. This leads to a discontinuity in the surface normal vector for those x such that $\phi(x) = 0$. Thus, if a straight line conic is desired, it must be managed by placing Q_1 on the line segment Q_0Q_2. If the shoulder curve is used in place of the shape function, the shoulder must also lie on the segment and ϕ computed by a limiting process.

It should be noted that region III is involved only in the case of the hyperbola. Since a continuous surface cannot involve both branches of a hyperbola, region III is excluded by the constraint $t < 1$ ($t = 1$ only in the case of a degenerate hyperbola). Another choice for a multiconic representation is whether to include the full conics in regions I and II, or only the segment in I. In the first instance the conics must be limited to ellipses; otherwise the surface has infinite extent. The limitation to region I requires the constraint $0 \leq t < 1$. If the multiconic is so restricted, another multiconic of similar type can be adjoined along Q_0 or Q_2 with C_1 continuity if and only if Q_1 is implicitly defined by the functions S_0 and S_2 mentioned in the Introduction.

The equations for the multiconic surface are obtained by inserting the argument x appropriately in (1), (2), and (3),

$$\xi'(x) = s(1-t)\xi_2'(x) + t\xi_1'(x), \tag{9}$$

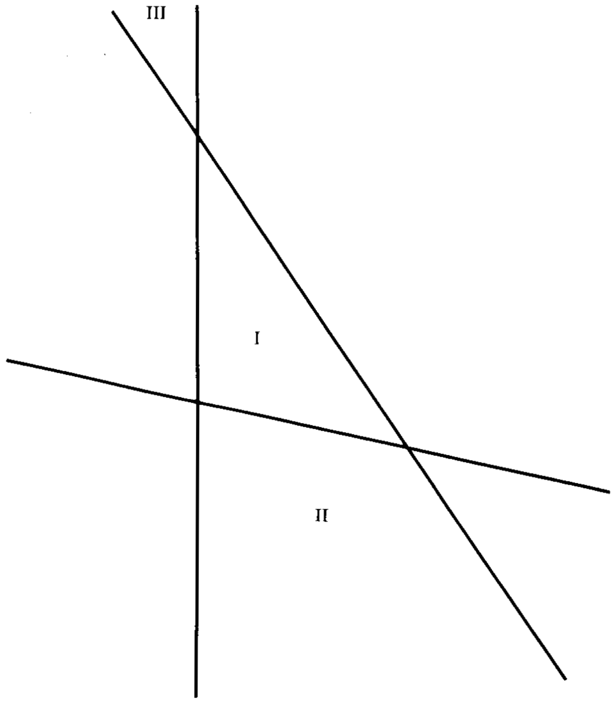


Figure 3 Plane regions determined by conic parameters (s, t).

$$\text{or } \xi(x) = s(1-t)\xi_2(x) + t\xi_1(x) + (1-s)(1-t)\xi_0(x), \tag{10}$$

together with

$$g(x, s, t) = t^2 - s(1-s)(1-t)^2\,\phi(x) = 0. \tag{11}$$

It is evident that the form of the representation is invariant under translation and also under rotation of coordinates about the x axis.

Line intersections with a multiconic

Let the line equation be $\eta = a + bu$, where $\eta = (x, y, z)$, $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ and u is a parameter.

For the case $t < 1$, define the interior of the multiconic to be the convex region delimited by the complete ellipse for each slice.

If $b_1 \neq 0$, select a set of argument values $\{x_i\}$, and compute the corresponding values $\{y_i, z_i\}$ from the line equation. Excluding the exceptional cases noted in the previous section compute $\{s_i, t_i\}$ from Eq. (10), hence $\{g(x_i, s_i, t_i)\}$ using Eq. (11). Now $g(x, s, t) < 0$ inside the convex region and $g(x, s, t) > 0$ outside. If any $g(x_i, s_i, t_i) = 0$, then x_i, y_i, z_i is a point of intersection. If g changes sign for two consecutive x_i, there exists a point of intersection for some x between the two values. This value, and hence the point of intersection, can be easily computed using some form of *regula falsi* [6, 7, 8].

For the exceptional cases, including the case $b_1 = 0$, special procedures must be devised.

Reasonable choices of the test set $\{x_i\}$ depend to some extent on the nature of the defining functions, as do techniques for excluding those arguments that are not feasible. Spline conics, for example, lend themselves to bounding segments of the surface in such a way that entire segments can be rapidly excluded.

For the case $0 \leq t < 1$, some intersections may occur on a part of the conic that is not contained in the defining triangle. These are false intersections and must be excluded.

Plane intersection with a multiconic surface

Consider a plane $\eta = A + Bv + Cw$, where $\eta = (x, y, z)$; A, B, and C are constant vectors for which B_1 and C_1 are not both zero; and v and w are parameters. Again choose a set $\{x_i\}$. The plane intersects the plane $x = x_i$ for a specific x_i in a line $\xi = a + bu$, where $\xi = (y, z)$, a and b are constant vectors. In translated coordinates, $\xi' = a' + bu$.

The problem now is to determine its intersections with the conic, i.e., to solve simultaneously

$$\xi_2{}'s(1-t) + \xi_1{}'t = a_1{}' + bu,$$
$$t^2 - \phi s(1-s)(1-t)^2 = 0. \tag{12}$$

When u is eliminated, the system becomes

$$ps(1-t) + qt = r,$$
$$t^2 - \phi s(1-s)(1-t)^2 = 0, \tag{13}$$
$$r = a_1{}'b_2 - a_2{}'b_1,$$

where

$$p = y_2{}'b_2 - z_2{}'b_1,$$
$$q = y_1{}'b_2 - z_1{}'b_1,$$
$$r = a_1{}'b_2 - a_2{}'b_1. \tag{14}$$

Let

$$w^2 = \phi s(1-s), \; t = w/(1+w).$$

Define

$$P^2 = \phi(r-q)^2 + 4r(p-r), \tag{15}$$

where P represents the positive square root if $P^2 > 0$; also

$$R = p^2 + \phi(r-q)^2,$$
$$F = \phi(r-q)(p-2r) + ep\sqrt{\phi}\,P,$$
$$G = 2p^2 + \phi(r-q)(p-2q) + ep\sqrt{\phi}\,P,$$
$$H = 2pr + \phi(r-q)^2 + e(r-q)\sqrt{\phi}\,P, \tag{16}$$

where $e = \pm 1$.

Certainly $R \geq 0$. Assume for the moment that $R > 0$, $P^2 \geq 0$ and note that $G = 2R + F$, so that $G > F$. Let $s = H/2R$, $w = F/2R$ and, if $F + 2R \neq 0$, $t = F/(F + 2R) = F/G$. With these values, Eqs. (13) are satisfied.

It can be shown that $0 \leq H \leq 2R$, whence $0 \leq s \leq 1$, as is evident geometrically. The point corresponding to (s, t) is in region I if and only if $0 \leq t < 1$ and is in regions I and II if and only if $t < 1$. For region I the condition becomes $0 \leq F/G < 1$, or $0 \leq F < G$ if $G > 0$. For $G < 0$ it becomes $0 \geq F > G$, which is not possible. Hence the condition for region I is $0 \leq F < G$. By a similar argument the condition for regions I and II is $F < G$, $G > 0$.

Now consider the case $R = 0$. Only two possibilities arise. One of these is $p = 0$, $\phi = 0$, which is excluded; the other is $p = 0$, $r = q$. In both cases $R = F = G = H = 0$ and $P^2 = -4r^2$. Unless $r = 0$ no solutions exist. If $p = q = r = 0$ then the pair of equations reduce to $0 = 0$, $t^2 - \phi s(1-s)(1-t)^2 = 0$, whose solutions are

$$t = w/(1+w), \; w = \pm\sqrt{\phi s(1-s)} \qquad 0 \leq s \leq 1. \tag{17}$$

Points in region I are obtained for $w = +\sqrt{\phi s(1-s)}$, in region II for $w = -\sqrt{\phi s(1-s)}$.

In summary (17) solves the problem for $R = r = 0$, and no solutions exist if $R = 0$, $r \neq 0$. For numerical calculation, the criterion for the existence of this case is $R \leq \varepsilon$, $|r| \leq \varepsilon$ for a preassigned small number ε.

The algorithm leads to 0, 1, or 2 solutions for each value of x_i. Proceeding through successive values of the x_i produces at most two strings of points belonging to the intersection. Whenever there are two strings, membership in one string or the other is determined by the value of e in the preceding algorithm. More information concerning the organization of the strings can be obtained by calculating the intersection of the plane with the boundary multiconic curves $Q_0(x)$, $Q_2(x)$ and the initial and terminal conics. In addition values of $x = x^*$ for which $P = 0$ with s and t valid are significant. In such cases the line of intersection of the given plane and the plane $x = x^*$ is tangent to the conic at $x = x^*$ and also to the curve of intersection of the plane and the multiconic.

The normal vector

The current practice in dealing with polyconics has been to use a set of points in the neighborhood of the point of interest to approximate the unit vector. This vector can be computed directly from the defining equations.

Given the set of functions

$$y = \lambda(x, s, t),$$
$$z = \tau(x, s, t),$$
$$\sigma(x, s, t) = 0, \tag{18}$$

it is not difficult to show that

$$\left.\frac{\partial z}{\partial y}\right|_{x=\text{constant}} = \begin{vmatrix} \tau_s & \tau_t \\ \sigma_s & \sigma_t \end{vmatrix} \div \begin{vmatrix} \lambda_s & \lambda_t \\ \sigma_s & \sigma_t \end{vmatrix}, \qquad (19)$$

$$\left.\frac{\partial z}{\partial x}\right|_{y=\text{constant}} = -\begin{vmatrix} \lambda_x & \lambda_s & \lambda_t \\ \tau_x & \tau_s & \tau_t \\ \sigma_x & \sigma_s & \sigma_t \end{vmatrix} \div \begin{vmatrix} \lambda_s & \lambda_t \\ \sigma_s & \sigma_t \end{vmatrix}, \qquad (20)$$

provided that the denominator is not zero, and the various functions satisfy appropriate continuity and differentiability conditions.

The components of a unit normal vector are proportional to

$$\begin{vmatrix} \lambda_x & \lambda_s & \lambda_t \\ \tau_x & \tau_s & \tau_t \\ \sigma_x & \sigma_s & \sigma_t \end{vmatrix} : -\begin{vmatrix} \tau_s & \tau_t \\ \sigma_s & \sigma_t \end{vmatrix} : \begin{vmatrix} \lambda_s & \lambda_t \\ \sigma_s & \sigma_t \end{vmatrix} \qquad (21)$$

whether or not the last term is zero.

In the present case

$$\lambda(x, s, t) = y_2(x)s(1-t) + y_1(x)t + y_0(x)(1-s)(1-t),$$

$$\tau(x, s, t) = z_2(x)s(1-t) + z_1(x)t + z_0(x)(1-s)(1-t),$$

$$\sigma(x, s, t) = t^2 - \phi(x)s(1-s)(1-t)^2. \qquad (22)$$

For these functions it follows that

$$\begin{vmatrix} \tau_s & \tau_t \\ \sigma_s & \sigma_t \end{vmatrix} = (1-t)B, \quad \begin{vmatrix} \lambda_s & \lambda_t \\ \sigma_s & \sigma_t \end{vmatrix} = (1-t)C,$$

$$\begin{vmatrix} \lambda_s & \lambda_t \\ t_s & t_t \end{vmatrix} = 2E(1-t),$$

$$B = z_2'(x)[2t + \phi(x)s(1-t)] + z_1'(x)\phi(x)(1-2s)(1-t),$$

$$C = y_2'(x)\ [2t + \phi(x)s(1-t)] + y_1'(x)\phi(x)(1-2s)(1-t),$$

$$E = (y_2'(x)z_1'(x) - y_1'(x)z_2'(x))/2. \qquad (23)$$

A unit normal therefore is proportional to

$$\lambda_x B - \tau_x C + 2\sigma_x E : -B : 2E, \qquad (24)$$

where

$$\lambda_x = \dot{y}_2(x)s(1-t) + \dot{y}_1(x)t + \dot{y}_0(x)(1-s)(1-t),$$

$$\tau_x = \dot{z}_2(x)s(1-t) + \dot{z}_1(x)t + \dot{z}_0(x)(1-s)(1-t),$$

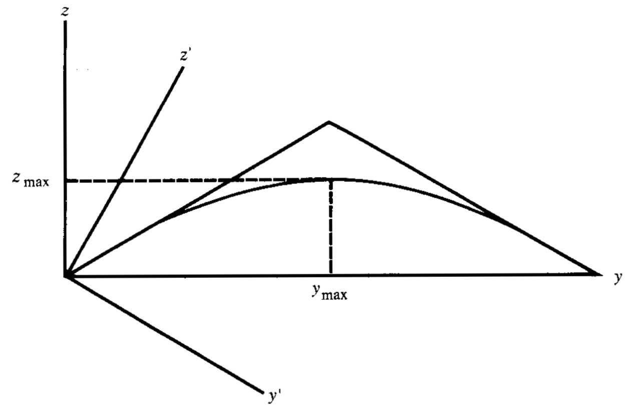


Figure 4 Conic section after translation and rotation of coordinates.

$$\sigma_x = -\dot{\phi}(x)s(1-s)(1-t)^2. \qquad (25)$$

The dot notation indicates differentiation with respect to the independent variable.

The possibility exists that the three components all reduce to zero. It is not difficult to show that, at any value of x for which this condition exists, either the conic reduces to a point, or the conic in question is a parabola and the circumstance occurs only at the parabolic point at infinity, which is of no interest here.

Values of x at which control curves have infinite slope need special attention. Except for the unlikely case that the first component remains finite, the unit normal vector is $\pm(1, 0, 0)$. The sign can be chosen in a consistent way by computing the normal at a nearby point.

Volume

In this section the volume contained in a region bounded by a multiconic surface and the ruled surface defined by the set of triangle bases is obtained for the domain (x_0, x_1). The procedure is to sum the infinitesimals $A(x)dx$ where $A(x)$ is the area of the conic section at x contained in region I. In order to compute that area it is convenient to translate the origin to Q_0, and rotate coordinates so that Q_2 lies on the new y axis. Then $A = \int_0^{z_{\max}} (y_+ - y_-)\,dz$.

In Fig. 4, y_+ refers to values of y greater than $y_{\max}$, y_- to values of y less than $y_{\max}$, the two values of y being taken for a fixed value of z. It is convenient here for notational purposes to let y, z represent the rotated coordinates rather than the original coordinates, which for the moment will be called y'', z''. Let $y' = y'' - y_0''$, $z' = z'' - z_0''$, $r = \sqrt{(y_2')^2 + (z_2')^2}$, $C = y_1'y_2' + z_1'z_2'$. As before $E = (y_2'z_1' - y_1'z_2')/2$. Consider the transformation

$$y = (y_2'y' + z_2'z')/r,$$

$$z = \text{sign}\ E(-z_2'y' + y_2'z')/r. \qquad (26)$$

It is clear that the transformation is a rotation if $E > 0$ and a rotation and a reflection if $E < 0$. If $r = 0$ or $E = 0$ the conic is a straight line or a point, with area 0, which case is automatically treated in the following.

Because a rotated (and reflected) conic remains a conic, and $y_1 = C/r$, $y_2 = r$, $z_1 = 2|E|/r$, $z_2 = 0$, it follows that in the new coordinate system the equation of the conic is

$$y = rs(1-t) + Ct/r,$$

$$z = 2|E|t/r. \qquad (27)$$

With

$$t = w/(1+w),\; w = \sqrt{\phi s(1-s)}$$

and

$$dt/dw = 1/(1+w)^2,$$

the maximum value of t occurs at the maximum value of w, i.e., at $s = 1/2$. But z is maximum when t is maximum, hence

$$w_{\max} = \sqrt{\phi}/2,\; t_{\max} = \sqrt{\phi}/(2+\sqrt{\phi}) = \rho,$$

$$z_{\max} = 2|E|\rho/r. \qquad (28)$$

In terms of the shape function ρ, the solution of s as a function of ρ leads to

$$s_+ = (\rho + \sqrt{\rho^2 - w^2(1-\rho)^2})/2\rho,$$

$$s_- = (\rho - \sqrt{\rho^2 - w^2(1-\rho)^2})/2\rho. \qquad (29)$$

Now

$$y_+ = r\,s_+(1-t) + Ct/r,$$

$$y_- = r\,s_-(1-t) + Ct/r, \qquad (30)$$

and therefore

$$\begin{aligned} y_+ - y_- &= r(s_+ - s_-)(1-t) \\ &= r\sqrt{\rho^2 - w^2(1-\rho)^2}\,(1-t)/\rho \\ &= r\sqrt{\rho^2(1-t)^2 - (1-\rho)^2t^2}/\rho. \end{aligned} \qquad (31)$$

Because $dz/dt = 2|E|/r$, it follows that

$$A = (2|E|/\rho)\int_0^\rho \sqrt{\rho^2(1-t)^2 - (1-\rho)^2t^2}\,dt. \qquad (32)$$

The integral can be evaluated as follows:

- $\rho < 1/2$

$$A = \frac{|E|\rho(1-\rho)^2}{(1-2\rho)^{3/2}}\left[\frac{\pi}{2} - \arcsin\frac{\rho}{1-\rho} - \frac{\rho\sqrt{1-2\rho}}{(1-\rho)^2}\right]. \qquad (33)$$

- $\rho = 1/2$

$$A = 2|E|/3.$$

- $\frac{1}{2} < \rho \le 1$

$$A = \frac{|E|\rho(1-\rho)^2}{(2\rho-1)^{3/2}}\left[\frac{\rho\sqrt{2\rho-1}}{(1-\rho)^2} - \log\frac{\rho+\sqrt{2\rho-1}}{1-\rho}\right]. \qquad (34)$$

Hence the volume is

$$V = \int_{x_0}^{x_1} |E(x)|\; F[\rho(x)]\,dx, \qquad (35)$$

where

$$\begin{aligned} F(\rho) &= \frac{\rho(1-\rho)^2}{(1-2\rho)^{3/2}}\left[\frac{\pi}{2} - \arcsin\frac{\rho}{1-\rho} - \frac{\rho\sqrt{1-2\rho}}{(1-\rho)^2}\right], && 0 \le \rho < 1/2 \\ &= 2/3, && \rho = 1/2 \\ &= \frac{\rho(1-\rho)^2}{(2\rho-1)^{3/2}}\left[\frac{\rho\sqrt{2\rho-1}}{(1-\rho)^2} - \log\frac{\rho+\sqrt{2\rho-1}}{1-\rho}\right], && 1/2 < \rho \le 1. \end{aligned} \qquad (36)$$

Obviously, to evaluate this volume, the zeros of $E(x)$ must be known. From the design point of view this implies x slices in which the conic reduces to a straight line or to a point.

If $\rho < 1/2$ the result for the total volume in domains I and II is

$$V = \int_{x_0}^{x_1} |E(x)|\; F_1[\rho(x)]\,dx, \qquad (37)$$

where $F_1(\rho) = \pi\rho(1-\rho)^2/(1-2\rho)^{3/2}$. This, of course, applies only to multiconics for which $\phi < 4$, or $\rho < 1/2$.

Surface area

The surface area of a multiconic can be put in the form [9]

$$A = \int_{x_0}^{x_1}\int_0^1 \sqrt{g(x,s)}\; ds\,dx, \qquad (38)$$

where $g(x,s) = (y_xz_s - y_sz_x)^2 + (y_s)^2 + (z_s)^2$, where the partial derivatives y_x, z_x are computed for s constant, and y_s and z_s are computed for x constant. Now

$$\xi_s = \xi_2'(1-t) + (\xi_1' - \xi_2's)t_s,$$

$$t_s = \phi(1-2s)/2w(1+w)^2, \qquad (39)$$

where $w^2 = \phi s(1-s)$.

Unfortunately t_s is not finite at $s = 0$ and $s = 1$. The difficulty can be removed by a change of variable

$$s = u^2/[u^2 + (1-u)^2], \qquad (40)$$

whence

$$t = 2\rho u(1-u)/[2u(1-u) + (1-\rho)(2u-1)^2],$$

$$w^2 = \phi u^2(1-u)^2/[u^2 + (1-u)^2]^2. \qquad (41)$$

Then

$$A = \int_{x_0}^{x_1} \int_0^1 \sqrt{g(x, u)}\, du dx, \qquad (42)$$

and

$$\xi_x = \dot{\xi}_2' s(1-t) + \dot{\xi}_1' t + \dot{\xi}_0 + (\xi_1' - \xi_2' s) t_x,$$
$$t_x = \dot{\phi}\, w/2\phi(1+w)^2,$$
$$\xi_u = \xi_2'(1-t)\, s_u + (\xi_1' - \xi_2' s) t_u,$$
$$s_u = 2u(1-u)/[u^2 + (u-1)^2]^2,$$
$$t_u = 2\rho(1-\rho)(1-2u)/[2u(1-u)$$
$$+ (1-\rho)(1-2u)^2]^2. \qquad (43)$$

Except for the case where control curves have an infinite slope, the double integral can be evaluated numerically. Otherwise further analysis is required.

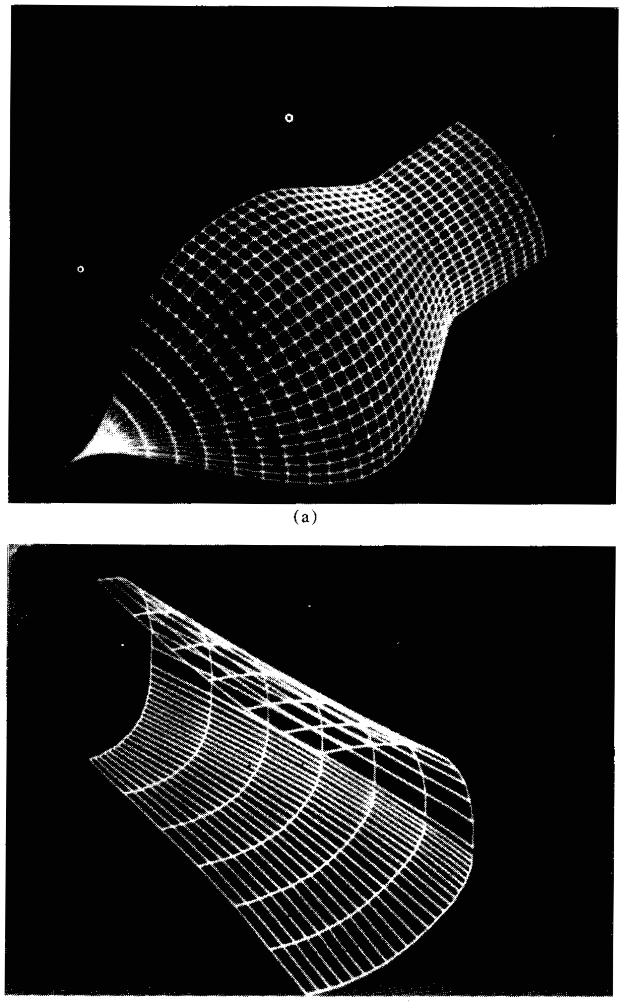
Note the surface area for the surface in region I is obtained when w is taken as the positive square root, and in region II when w is taken as the negative square root.

Conclusion and examples

Based on the preceding algorithms, a flexible system can be designed so that new classes of functions $Q_i(x)$, $\phi(x)$, $\rho(x)$ can be introduced as desired. This requires for the most part only the addition of appropriate function subroutines for the function calls of the algorithms. A major advantage of the system is the fact that there is no confusion about points belonging to a complete conic but not belonging to the segment which is part of the surface.

As for speed and efficiency, limited experience to date for spline conic surfaces suggests that there is conservatively an order of magnitude improvement in design time and a five-to-one ratio for numerical control processing time. Figure 5(a) exhibits a spline conic which could not be designed by previous methods unless split into three or more surfaces. Moreover the design required less input data than normally required for standard polyconic input.

Figure 5(b) is one-half of a fuselage section designed by adjoining four spline conics. The other half of the section is a mirror image of this part. It was estimated that designing this segment using polyconics would take four to six man weeks. Subsequently, using spline conics, the actual time required was ten man hours, of which half was expended in learning to use the system.



(a)

(b)

Figure 5 Design by spline conics. (a) Example 1. (b) Example 2. Reproduced with the permission of Teledyne Ryan Aeronautical, San Diego, California.

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