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Numerical Calculation of Normal Modes for Underwater Sound Propagation

Abstract: Hartree's method for calculation of atomic wave functions is applied to the Schroedinger-like normal mode equation for underwater sound propagation. Rapid convergence was obtained for the twelve normal modes at five Hertz with a typical velocity profile. The normal modes are given, along with an example of the pressure field, and a means for numerical calculation of the near field modes is suggested.

Introduction

The ray theory for propagation of underwater sound becomes inapplicable when the wave length is commensurate with the spatial fluctuations of the sound speed. In this case the sound pressure field propagating from a single-frequency point source can be found from the wave equation if one knows the sound velocity and density of the medium throughout the region of interest, as well as the location of the region's boundaries. This problem can be solved numerically by finite difference methods [1]. One would like, however, to find closed-form solutions to the propagation equations. If found, they usually offer improved insight into the physics of the processes involved.

Much work along these lines has been done. Epstein [2] found closed-form hypergeometric solutions to a particular form of velocity profile having no lower boundary. Pekeris [3] solved the propagation problem for a constant velocity profile with several discontinuities. Other solutions have been found following previous work in radio wave propagation [4].

A useful formal solution to the problem has been found in terms of the normal modes [2, 4 – 12]. If the velocity profile is a function only of depth, the wave equation is separable into functions of depth and functions of range. The functions of range are the well-known Hankel functions. If one can ignore the near field, the functions of depth are the normal mode functions, which are solutions to a Schroedinger-like equation. One need, then, only solve a one-dimensional differential equation similar to that for the forced vibrations of a nonuniformly suspended string.

Both exact and approximate solutions for the normal modes have been obtained [2, 4, 6, 13]. Pedersen and Gordon [7] use Airy functions, which are solutions for a piecewise linear velocity profile. Nicholas [10] uses Hermite functions for a parabolic profile and applies perturbation theory to approximate solutions for other profiles. Williams [11,12] solves for a constant velocity profile and uses perturbation theory to obtain approximate solutions for other profiles.

The approach in this paper is to solve the Schroedinger-like normal mode equation with a method used by Hartree and others [14 – 17] to calculate atomic wave functions. With this shooting method one can compute, with relative ease, the normal modes to any desired accuracy for any depth-dependent velocity profile. It is possible to include density in the equations so that the characteristics of the bottom can be more accurately modeled. No formal comparison was made between this and alternative methods of computing sound pressure fields. However, information on the accuracy and convergence properties is presented to give other researchers an idea of the appropriateness of the method for their applications.

Basic equations

The homogeneous sound wave equation, in a region with no source, is given by [3 – 6,10,13]

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0, \qquad (1)$$

where ϕ is the velocity potential, t is time, ∇^2 is the Laplacian operator, and c is the sound velocity. The displacement rate $\partial \mathbf{d}/\partial t$ and pressure p are related to ϕ by

$$\frac{\partial \mathbf{d}}{\partial t} = -\nabla\phi \qquad (2)$$

and

$$p = \rho \frac{\partial \phi}{\partial t}, \qquad (3)$$

where $\mathbf{d}$ is the displacement vector, ∇ is the gradient operator, and ρ is the density, which is constant except for possible discontinuities. At any boundary for which c or ρ is discontinous, the pressure must be continuous; i.e.,

$$\rho_1 \frac{\partial \phi_1}{\partial t} = \rho_2 \frac{\partial \phi_2}{\partial t}, \qquad (4)$$

and the velocity normal to the boundary must be continuous; i.e.,

$$\mathbf{n} \cdot \nabla\phi_1 = \mathbf{n} \cdot \nabla\phi_2, \qquad (5)$$

where the subscripts 1 and 2 indicate the two sides of the boundary and $\mathbf{n}$ is a vector normal to the boundary.

The field near a simple harmonic point source is given by [3–6,13]

$$\phi(r, z, t) = \frac{S}{[r^2 + (z - z_0)^2]^{\frac{1}{2}}} \times \sqrt{2} \cos \omega\{t - \frac{1}{c(z_0)} [r^2 + (z - z_0)^2]^{\frac{1}{2}}\}, \qquad (6)$$

where r and z are cylindrical coordinates, c is assumed to be a function only of z, ω is angular frequency, and the source coordinates are $(r, z) = (0, z_0)$. The coefficient S is, from (3) and (6),

$$S = \frac{R[\overline{p(R, z_0)^2}]^{\frac{1}{2}}}{\omega\rho(z_0)}, \qquad (7)$$

where R is some distance close to the source, the bar indicates a time average, and ρ is a function only of z. One chooses the cylindrical system with the origin at the water surface so that z represents depth and r represents range.

The point source expression (6) satisfies the inhomogeneous equation [2,5,10,12]

$$\left[\nabla^2 - \frac{1}{c(z)^2} \frac{\partial^2}{\partial t^2}\right] \phi(r, z, t) = -2S \frac{\delta(r)}{r} \delta(z - z_0) \sqrt{2} \cos \omega t, \qquad (8)$$

where $\delta(r)$ and $\delta(z - z_0)$ are Dirac delta functions. The object of subsequent analysis is to find a solution to (8) that matches the required boundary conditions.

The sound field $\phi(r, z, t)$ can be represented conveniently by the complex spatial function $\psi(r, z)$ that appears in the expression

$$\phi(r, z, t) = \mathrm{Re}\ [\sqrt{2}\, S\, \psi(r, z) \exp(-i\omega t)], \qquad (9)$$

where Re indicates that the real part is to be taken. The field near a simple harmonic point source is

$$\psi(r, z) = \frac{\exp\left\{i \frac{\omega}{c(z_0)} [r^2 + (z - z_0)^2]^{\frac{1}{2}}\right\}}{[r^2 + (z - z_0)^2]^{\frac{1}{2}}}, \qquad (10)$$

and the equation to be solved becomes

$$\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} + k^2(z)\right] \psi(r, z) = -2 \frac{\delta(r)}{r} \delta(z - z_0), \qquad (11)$$

where the ∇^2 operator has been expressed in cylindrical coordinates and

$$k(z) = \omega / c(z). \qquad (12)$$

Let $v(\alpha, z)$ be the Bessel transform [6,18] of $\psi(r, z)$ such that

$$\psi(r, z) = \int_0^\infty v(\alpha, z)\, J_0(\alpha r)\alpha\, d\alpha. \qquad (13)$$

When one puts (13) into (11) he obtains

$$\int_0^\infty \left\{\left[-\alpha^2 + \frac{\partial^2}{\partial z^2} + k^2(z)\right] v(\alpha, z)\right\} J_0(\alpha r)\alpha\, d\alpha = -2 \frac{\delta(r)}{r} \delta(z - z_0). \qquad (14)$$

Because the Bessel transform implies

$$F(\alpha') = \int_0^\infty \left[\int_0^\infty F(\alpha)\, J_0(\alpha r)\alpha\, d\alpha\right] J_0(\alpha' r) r\, dr, \qquad (15)$$

one may multiply (14) by $J_0(\alpha' r) r\, dr$ and integrate to get

$$\left[-\alpha^2 + \frac{d^2}{dz^2} + k^2(z)\right] v(\alpha, z) = -2\, \delta(z - z_0), \qquad (16)$$

which is an equation for the forced vibration of a nonuniformly suspended string.

Let v be the sum of a continuous part w and a discrete part u:

$$v(\alpha, z) = w(\alpha, z) + \sum_n a_n(\alpha) u_n(z). \qquad (17)$$

Let $u_n(z)$ be the eigenfunctions of

$$\left[\frac{d^2}{dz^2} + k^2(z)\right] u_n(z) = \gamma_n^{\,2} u_n(z), \qquad (18)$$

which is a Schroedinger equation. One can require

$$\int_0^\infty u_n(z) u_{n'}(z) dz = \delta_{nn'} \qquad (19)$$

and

$$\int_0^\infty u_n(z)\left[-\alpha^2 + \frac{\partial^2}{\partial z^2} + k^2(z)\right] w(\alpha, z)\,dz = 0, \qquad (20)$$

because the $u_n(z)$ are discrete eigenfunctions of the left side of (16) and the $w(\alpha, z)$ can be expressed in terms of orthogonal continuous eigenfunctions of the left side of (16). Now, putting (17) into (16), multiplying by $u_n(z)$, and intergrating over z from 0 to ∞, one gets

$$a_n(\alpha)(-\alpha^2 + \gamma_n^{\;2}) = -2\, u_n(z_0), \qquad (21)$$

which, when solved for $a_n(\alpha)$ and used in (17), gives

$$v(\alpha, z) = w(\alpha, z) + 2\sum_n \frac{u_n(z_0)}{\alpha^2 - \gamma_n^{\;2}}\, u_n(z). \qquad (22)$$

Putting this into (16), one obtains the following equation for the continous part:

$$\left[\frac{d^2}{dz^2} + k^2(z) - \alpha^2\right] w(\alpha, z)$$
$$= -2\,\delta(z - z_0) + 2\sum_n u_n(z_0) u_n(z). \qquad (23)$$

When one solves (18) and (23) for u and w, he may put them into (17) and (13) to obtain

$$\psi(r, z) = \int_0^\infty w(\alpha, z) J_0(\alpha r)\alpha\, d\alpha$$
$$+ 2\sum_n \int_0^\infty \frac{J_0(\alpha r)\alpha\, d\alpha}{\alpha^2 - \gamma_n^{\;2}}\, u_n(z_0) u_n(z). \qquad (24)$$

Now, because $J_0(\alpha r) = \frac{1}{2}[H_0^{\,(1)}(\alpha r) - H_0^{\,(1)}(-\alpha r)]$, where $H_0^{\,(1)}(\alpha r)$ is a Hankel function, one may write

$$2\int_0^\infty \frac{J_0(\alpha r)\alpha\, d\alpha}{\alpha^2 - \gamma_n^{\;2}} = \int_{-\infty}^\infty \frac{H_0^{\,(1)}(\alpha r)\alpha\, d\alpha}{\alpha^2 - \gamma_n^{\;2}}$$
$$= i\,\pi H_0^{\,(1)}(\gamma_n r), \qquad (25)$$

where the path of integration was taken to be consistent with the radiation condition [4,6,12,13,18], and $H_0^{\,(1)}(i\infty) = 0$. The final solution is, therefore [4–7,9,10,12,13,19],

$$\psi(r, z) = \int_0^\infty w(\alpha, z)\, J_0(\alpha r)\alpha\, d\alpha$$
$$+ i\pi \sum_n H_0^{\,(1)}(\gamma_n r) u_n(z_0) u_n(z). \qquad (26)$$

In summary, the pressure field is implied by (3), (7) and (9). One first solves the eigenvalue problem (18) for γ_n and $u_n(z)$ by a conventional method, such as the shooting method. This gives the discrete part of ψ in (26). The continuous part can usually be ignored in the far field [11–13]. If one wishes to calculate the continuous part, however, he may do so by numerically integrating (26) over α and (23) over z.

Shooting method

The eigenfunctions $u_n(z)$ of (18) are solutions to the Schroedinger-like equation

$$\left[\frac{d^2}{dz^2} + k^2(z)\right] u(z) = \gamma^2 u(z). \qquad (27)$$

This equation can be solved numerically with a shooting method which has been used successfully [14–17] to calculate atomic wave functions.

The boundary conditions on $u(z)$ are given by (4) and (5):

$$\rho_1 u_1 = \rho_2 u_2 \qquad (28)$$

and

$$\frac{du_1}{dz} = \frac{du_2}{dz}. \qquad (29)$$

Although it is not necessary to do so, it will be assumed that ρ is constant over the semi-infinite interval $0 < z < \infty$. At $z = 0$ (the surface), the pressure is assumed to be zero. This implies $u(0) = 0$ as a boundary condition. It is also required that $u(\infty) = 0$.

These two boundary conditions imply that solutions of (27) exist only for certain discrete values of γ^2. One tries different values of γ^2 in succession and numerically integrates (27) from the left (u_L) and from the right (u_R) until these two solutions and their derivatives match at a fitting point. This point z_F is chosen near the rightmost zero of $\gamma^2 - k^2(z)$. Because the starting values from the left and right are chosen to match the boundary conditions, one has solutions for $u(z)$ and γ^2.

One desires a variation in γ^2, i.e., $\delta(\gamma^2)$, to cause the quantity $(1/u)(du/dz)$ to match on the left and on the right of z_F. This may be stated in variational notation as follows:

$$\left(\frac{1}{u_L}\frac{du_L}{dz}\right)_F + \delta\left(\frac{1}{u_L}\frac{du_L}{dz}\right)_F = \left(\frac{1}{u_R}\frac{du_R}{dz}\right)_F + \delta\left(\frac{1}{u_R}\frac{du_R}{dz}\right)_F. \qquad (30)$$

First, take the variation of (27),

$$\frac{d^2}{dz^2}\,\delta u + k^2\delta u = u\delta(\gamma^2) + \gamma^2\delta u. \qquad (31)$$

Multiplying (31) by u, (27) by δu, and subtracting, one obtains

$$u^2\delta(\gamma^2) = u\,\frac{d^2}{dz^2}\,\delta u - \delta u\,\frac{d^2 u}{dz^2}$$
$$= \frac{d}{dz}\left(u\delta\,\frac{du}{dz} - \frac{du}{dz}\,\delta u\right). \qquad (32)$$

Now, integrating (32) on the left and taking note that $u_L(0) = 0$, one gets

$$\delta(\gamma^2)\int_0^{z_F} u_L^{\;2} dz = u_L^{\;2}(z_F)\,\delta\left(\frac{1}{u_L}\frac{du_L}{dz}\right)_F. \qquad (33)$$

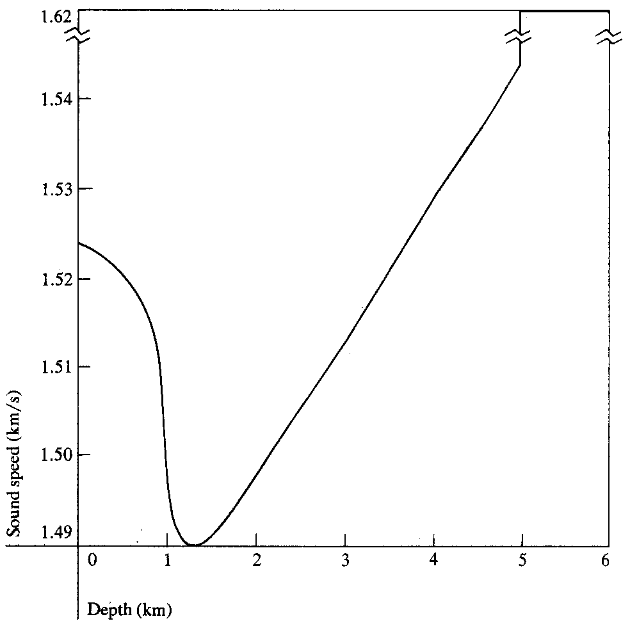

Figure 1 Sound speed profile.

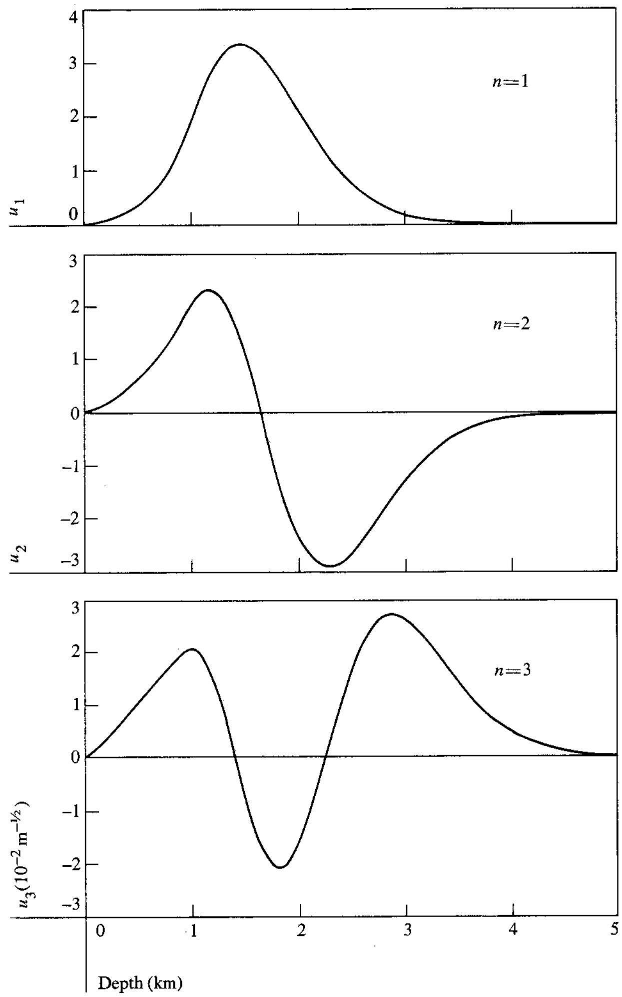

Figure 2 Normal mode functions n = 1,2,3.

Table 1 Eigenvalues γ_n of the twelve normal modes.

n	*Eigenvalue* (km^{-1})
1	20.97752000
2	20.81573370
3	20.68162562
4	20.56654163
5	20.46806597
6	20.37328392
7	20.26305241
8	20.12940776
9	19.97451506
10	19.80292468
11	19.61546118
12	19.41695439

Similarly, on the right, one obtains

$$\delta(\gamma^2) \int_{z_F}^{\infty} u_R^2 dz = -u_R^2(z_F)\delta\left(\frac{1}{u_R}\frac{du_R}{dz}\right)_F. \quad (34)$$

One can now solve (30), (33) and (34) for $\delta(\gamma^2)$:

$$\delta(\gamma^2) = \left[\left(\frac{1}{u_R}\frac{du_R}{dz}\right)_F - \left(\frac{1}{u_L}\frac{du_L}{dz}\right)_F\right] \Big/ \left[\frac{1}{u_L^2(z_F)}\int_0^{z_F} u_L^2 dz + \frac{1}{u_R^2(z_F)}\int_{z_F}^{\infty} u_R^2 dz\right], \quad (35)$$

which gives the change in γ^2 required to improve the match of $(1/u)(du/dz)$ at z_F. The integrals in (35) can be calculated by numerical integration at the same time as $u_L(z)$ and $u_R(z)$. They can also be used to normalize $u(z)$.

The infinite integral from z_F to ∞ can be handled in the following manner. Assume that the velocity is constant in the bottom boundary material, i.e., $k^2(z) = k_B^2$ for $z_F < z_B \le z < \infty$. Then (27) can be solved exactly in this region:

$$u_R(z) = u_R(z_B)\ \exp\ [-(\gamma^2 - k_B^2)^{\frac{1}{2}}(z - z_B)], \quad \gamma^2 > k_B^2. \quad (36)$$

Note that $u_R(\infty) = 0$ is satisfied. The integral in (35) can therefore be calculated with the aid of (36).

Results

The shooting method just described was used to calculate the normal mode functions $u_n(z)$, i.e., the eigenfunctions, for a sound speed profile given by Williams [12]. This profile is shown in Fig. 1. Note that the sound speed in the bottom is 1.62 km/s. One hundred points were taken from the speed curve and Eq. (27) was numerically integrated with a fourth-order Runge-Kutta method and a step size of 0.1 km at a frequency of 5 Hz.

The integrals in (35) were calculated from the left and from the right, as were $u_L(z)$ and $u_R(z)$, with Runge-Kutta integration. The starting values on the left were $u_L(0) = 0$ and $(du_L/dz)_{z=0} = 1$, the initial slope being arbitrary. The slope was normalized after each iteration. The starting values on the right were $u_R(z_B) = 1$ and $(du_R/dz)_{z=z_B}$ as determined by (36). The value of $u_R(z_B)$ was normalized after each iteration. The normalization of the function $u_n(z)$ satisfies (19) and was easy to perform using the integrals calculated for (35). Equation (35) gave fairly rapid convergence, i.e., 5 to 10 steps for the desired accuracy.

Twelve eigenfunctions $u_n(z)$, i.e., normal mode functions, and their associated eigenvalues γ_n were found. The eigenvalues are shown in Table 1. The first three eigenfunctions are also shown in Fig. 2. No more than twelve discrete modes are possible for this case.

A sample calculation of the sound pressure field was made with the source at a depth of 0.1 km, the field point at a depth of 1.2 km in the sound channel, and a range of 50 to 150 km. Equation (26) was used to perform the calculation. The continuous near-field term was ignored and the Hankel function was approximated with

$$H_0^{(1)}(\gamma_n r) \approx \sqrt{\frac{2}{\pi \gamma_n r}} \exp\left[i\left(\gamma_n r - \frac{\pi}{4}\right)\right]. \qquad (37)$$

The results are shown in Fig. 3.

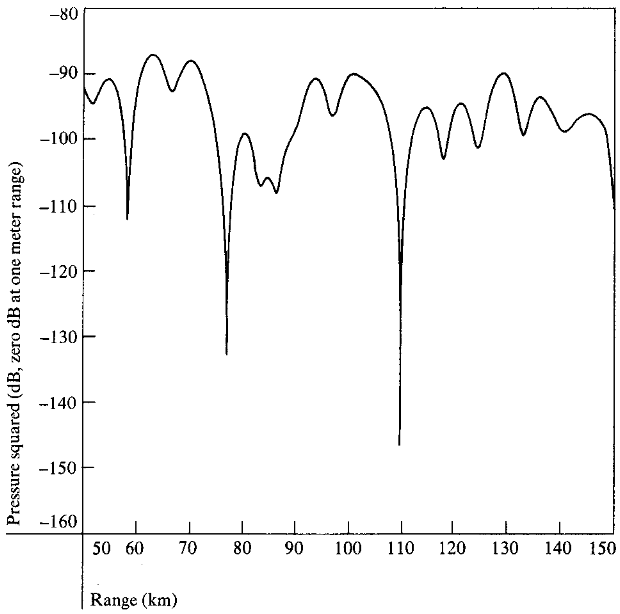

Figure 3 Sound pressure field at 5 Hz; source depth 0.1 km; field depth 1.2 km.

Accuracy

Given that there is no error in the integration of Eqs. (27) and (35), the difference in γ between successive iterations is a measure of error as well as of convergence [20]. This difference became less than 10^{-8} percent (ten place accuracy) and remained so for all modes.

The equations, however, were integrated numerically by the standard fourth-order Runge-Kutta method. An estimate of the integration error can be obtained by examining the size of the fifth-order term and its effects. This gave an error of less than 10^{-3} percent in u for the lower-order modes. Typically, for a Rayleigh quotient such as Eq. (35), the error in the eigenvalue is of second order if the error in the eigenvector is of first order. This implies an error in γ of less than 10^{-8} percent for the lower-order modes. For the twelfth and highest mode, the error was estimated to be less than about one percent for u and, correspondingly, less than 0.01 percent for γ.

A step size of 0.1 km was used for the integration, which implies a total of 50 steps for one iteration of each mode. A maximum of ten iterations per mode was required. Because there were twelve discrete modes, and twelve only, this gives a total of 6000 numerical integration steps for the entire calculation at 5 Hz. At higher frequencies one must decrease the step size proportionately as the minimum wave length (wave length of the highest order mode) decreases. The total number of discrete modes present is also proportional to the frequency. These two facts imply that the number of calculations increases with the square of the frequency, e.g., for 50 Hz, 600 000 integration steps would be required if one desired to calculate all of the approximately 120 modes.

Equation (35) could also be applied to piecewise exact solution methods [7], thus eliminating numerical integration error. The numerical approach, however, has the advantage that complex velocity profiles may be more easily specified. One does not have to conform the profile to a limited set of straight line regions, for example. Each intermediate integration step in such a region can have a different value of velocity.

One can improve the accuracy of the numerical approach by simply increasing the number of steps or by using more efficient integration techniques, e.g., variable step Runge-Kutta or Adams methods. In the application of Eq. (35) by numerical integration, no problem was encountered with convergence or the identification of modes, as was reported with Newton's method for finding eigenvalues in piecewise exact solutions [7].

Concluding remarks

Hartree's iterative shooting method was found to be useful in calculating normal mode eigenfunctions at low frequencies, e.g., 5 Hz. When the Runge-Kutta or a similar integration method is used, there is no limitation on the shape of the velocity profile. The profile can be described and the mode functions can be calculated as precisely as desired. It is possible to include density in the equations, so that the characteristics of the bottom can be more accurately modeled.

The sound field in Fig. 3 can be considered to be only representative of any real field. The errors in describing and measuring such quantities as the sound speed, even though small, can change the function in Fig. 3 considerably.

The curve does show what general properties one would expect to find in a real field. For example, it is possible for the sound energy received from a source to drop 60 dB in 5 km at 5 Hz. Such low energy points are extreme, but infrequent.

This method assumes the velocity to be a function of depth only. If one wishes to be more realistic and include range dependence, he must turn to other methods [1, 8].

Expressions for the numerical calculation of the near field modes appear in (23) and (26). Such calculations were not done in this study, but are relatively straightforward.

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