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Method for Estimation and Optimization of Printer Speed Based on Character Usage Statistics

Abstract: Many high-speed printers now in the field and under development use a constantly moving chain or train containing the characters required in the printing process. Generally they skip to the next line whenever the last character on a given line is printed. Since individual character usage varies widely, it may be possible to increase the printing speed by repeating high-usage characters more frequently on the chain than low usage ones.

This paper presents an analytic method of accurately estimating the printing speed of a chain printer for any character arrangement and describes a technique for determining the number of copies each character should have on the chain so that the printer will operate at or near maximum speed. Using these methods, significant increases in printing speeds have been obtained for actual applications.

Introduction

Many of the high-speed printers used as computer output devices employ a constantly moving "chain" containing the set of characters required in printing. For some of these chain printers, the chain itself may accommodate several copies of each character and the character arrangement may be arbitrary, though generally it is not. Printing speed for these printers depends both on the information to be printed and the character arrangement. Since most printing applications use some characters more often than others it may be possible to increase the printing speed by repeating high usage characters on the chain more frequently than low usage ones.

The purpose of this paper is two fold: (1) to present an analytic method of accurately estimating the printing speed of a chain printer for any character arrangement on a chain, given the character usage statistics and (2) to show how to improve chain character arrangements through the use of an iterative algorithm based on (1).

The paper is divided into four principal parts. Part I consists of background material on printers. The theory is developed for the analytic method of predicting printing speeds in Part II. Part III details the iterative algorithm for improving character arrangements. The theory and algorithm have been implemented in a flexible computer program which is available through SHARE under SDA 3542. Part IV presents results based on this program in which significant increases in printing speed have been obtained for actual applications (up to 26%). Calculated and observed printing speeds are in comfortable agreement.

Part I. Background

The pertinent features of a typical chain printer may be seen in Fig. 1. In our analysis, we assume that the printing operation is asynchronous. That is, after the recording medium (hereafter called paper) has been properly positioned for printing, the chain can be in any position relative to the paper. The actual character impressions on the paper are produced as the result of impacts between paper and chain characters due to the action of "hammers."

It should be pointed out that characters are not printed in the order in which they finally appear on the paper but in the order in which alignment occurs between chain characters and desired printing positions. Also, character spacing on the chain is generally not the same as the print position spacing. This serves to reduce or eliminate the number of possible simultaneous hammer firings, but in no way affects our analysis. The chain is assumed to move at constant speed and to be composed of characters of equal width. The time required for a character on the chain to advance one column will be taken as our unit of time and is called a "scan."

The time required to print a given character at a print position on the paper is determined by how long it takes the print chain to move so that a copy of the correct character on the chain is opposite to the print position. This time can vary from zero to the time it takes for the chain to move a distance equal to the maximum spacing between the leading edges of two successive copies of that character. Knowing the character arrangement on the chain and the initial position of the chain at the beginning of a new line, one can

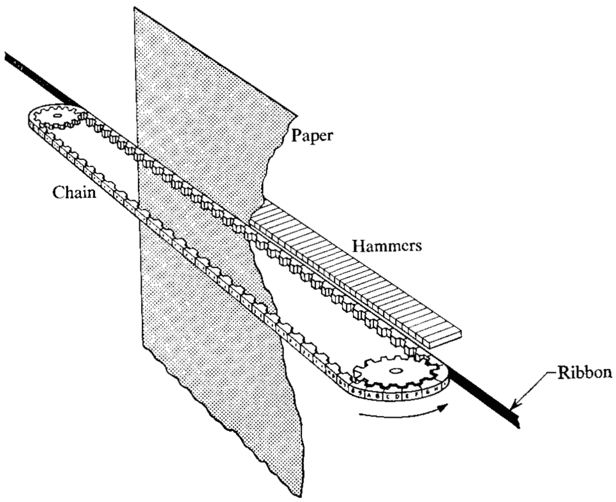
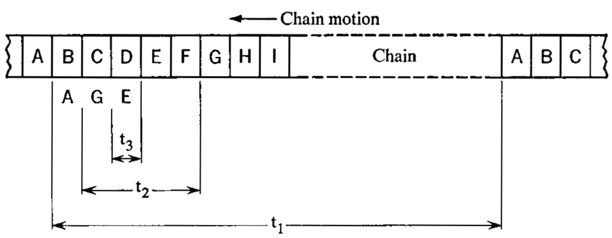


Figure 1 Chain printer elements.

Figure 2 Relation between time required to print a line and time for individual characters to reach print position. In the example shown, the letters AGE constitute an entire printed line. Hence, line print time = max $(t_1, t_2, t_3) = t_1$.



easily determine the print time for each character in a line. (See Fig. 2.) The time required to print the entire line is the maximum of the print times of the individual characters on the line.

In some printers, physical limitations demand that skipping to the next line not take place until some time interval T_0 has elapsed. Thus, no matter how quickly all characters are printed on a line, the effective print time cannot be less than T_0.

The usage of a character is defined to be the expected number of times a character occurs per line. It is estimated from any sample printing job by simply counting all the occurrences of a given character and dividing by the number of lines. The expected line length (expected number of characters per line) is estimated by summing the resulting averages, or more simply, its estimate is the total number of characters printed divided by the number of lines in the sample.

In 1960 D. N. Freeman performed an unpublished analysis in which he estimated the printing speed of a chain with all characters repeated the same number of times. The authors of this paper have not been able to find any published treatment of a general method for arranging characters on a chain to improve printer speed.

Part II. Formulas

• *General*

This part of the paper concerns the development of equations for cases in which the chain configuration (arrangement of characters on the chain) is already specified. An analytical expression is given relating chain configuration, character usage, and the per-line expected printing time. This provides a means for comparing different chain configurations which may be proposed for the same printer application.

The number of distinct characters appearing on the chain is denoted by n and the individual (distinct) characters are denoted by $C_1, \cdots, C_n$. Stated differently, the chain contains one or more copies of each of the characters $C_1, \cdots, C_n$. The number of copies of character C_i is denoted by r_i $(i = 1, \cdots, n)$ and the relationship

$$\sum_{i=1}^{n} r_i = Q$$

is assumed, defining Q as the chain length expressed in terms of the total number of character copies it contains. Thus, points on the moving chain return to their original positions every Q units of time. (Reference to Fig. 1 should make the notation clear.)

• *Character copies*

The individual copies of character C_i are denoted by C_{i1}, $C_{i2}, \cdots, C_{ir_i}$, the second subscript being explained as follows. Suppose any copy is chosen and denoted by C_{i1}. If the chain is moving and copy C_{i1} is opposite a given print position on a line, the remaining copies pass the same position in the order $C_{i2}, C_{i3}, \cdots, C_{ir_i}$. Of course, C_{ir_i} is followed by copy C_{i1} as the chain continues to move.

These and related remarks in subsequent paragraphs apply for $i = 1, \cdots, n$ unless specific contrary indications are made.

Fig. 3 illustrates the way in which several copies of the same character, C_i, might be arranged on the chain. It also shows how distances between successive copies are denoted. The distance between the leading edges of successive copies $C_{i(m-1)}$ and C_{im} is denoted by Q_{im}. Thus, the indicated distance between C_{i1} and C_{i2} is Q_{i2}; that between C_{i2} and C_{i3} is Q_{i3}; etc.; that between C_{ir_i} and C_{i1} is Q_{i1}. This implies

$$\sum_{j=1}^{r_i} Q_{ij} = Q\,,$$

where Q is the chain length as previously defined.

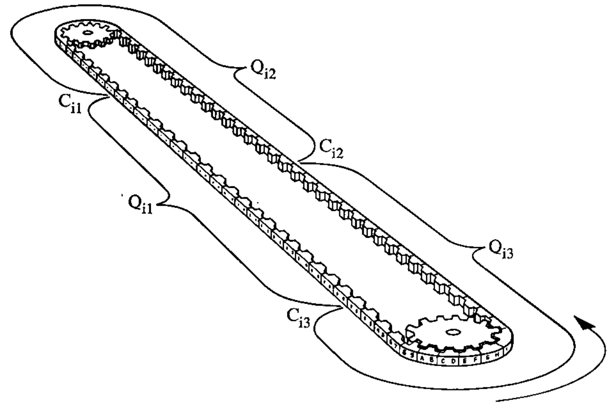


Figure 3 Definition of Q_{ij}, the distance between successive copies of character C_i on a chain of length Q copies. In the example shown $r_i = 3$ copies of C_i.

• *Direct consequences*

Now suppose C_{ij} is known to have been used for printing at a given position on a line; C_{ij} must have been the first among the copies of character C_i to reach the print position. Also, when printing of the line was initiated, the point position (i.e., an appropriate point associated with it) must have been located opposite a part of the chain immediately preceding the leading edge of C_{ij}. The eligible part of the chain for this is of length Q_{ij}.

Thus, if C_i is to be printed at a given position, the probability that copy C_{ij} will be used for the task is Q_{ij}/Q. Random chain orientation, as previously assumed, implies this.

The time required for printing at a given position must be between 0 and Q_{ij}, inclusive, whenever copy C_{ij} is used. This printing time is regarded as a random variable whose probability distribution is uniform on the interval 0 to Q_{ij}. Thus, the conditional probability, that when C_{ij} is used at a given position, the print time required does not exceed the value u (say), is u/Q_{ij} for $0 \leq u \leq Q_{ij}$.

Cases involving negative values of u are meaningless in the present context and receive no mention in discussions hereafter; cases involving $u \geq Q_{ij}$ will be treated as they arise.

• *Printed material*

Composition of printed material is considered next. Ideally, assumptions attendant to this matter should be very realistic. They must, at the same time, be amenable to mathematical development. Simultaneous satisfaction of both requirements is virtually impossible. Hence, the authors resort to assumptions which lead to reasonable approximations. This highlights the importance of results given in Part IV which indicate a satisfactory agreement between theory and observed printer performance in widely different applications.

Consider lines for which a maximum of M print positions may be used and suppose that, in printing, the number of print positions to be used on a line is regarded as a binomially distributed random variable with parameters p and M. That is to say, if W represents the number of characters printed on a line,

$$P(W = w) = C_w^M p^w q^{M-w}, \quad (w = 0, \cdots, M), \qquad (1)$$

in which $p, q > 0$ and $q = 1 - p$.

Further suppose that, if some character (unspecified) is to be printed at a given position, the probability that the character will be C_i is given by $\nu_i (i = 1, \cdots, n)$. The value of ν_i is assumed to be positive, constant and independent of print position.

If any position is to be left blank, the printing time required there is zero.

A final assumption is that printing time at any position is quite independent of what is to be printed at other positions on the line and the corresponding printing times there. (Conflicts between this assumption and realities of printing are recognized. However, these conflicts appear to have very little adverse effect on final results as judged from Part IV.)

• *Usage statistics and parameters*

Character usage statistics provide a base for estimating some parameters defined above. The average number of occurrences of characters $C_1, \cdots, C_n$ per line can be calculated from a sample as indicated in Part I. Those averages are denoted in Part III by $e_1, \cdots, e_n$ respectively and we can define

$$\bar{e} = \sum_{i=1}^{n} e_i .$$

Then $\bar{e}$ estimates the product Mp which is the expected number of characters printed on a line. The value of p is estimated by $\bar{e}/M$ and is the probability that any randomly chosen print position on a line will be used in printing (i.e., that the position will not be blank).

Similarly e_i/e is an estimate of ν_i $(i = 1, \cdots, n)$. A later section treats the case in which M and p vary in such a way that the product Mp tends to a constant λ. In that case the e_i values are estimates of the corresponding values $\lambda_i = \lambda\nu_i$ $(i = 1, \cdots, n)$ as defined in the referenced section.

Theory development, however, is based upon the assumption that all parameter values are known in advance.

• *Printing time U for a line*

Consider the printing time at a given position when the character to be printed is not known in advance, i.e., when the extent of prior knowledge is that the position will not be blank. If the time required is to exceed u, only copies whose Q_{ij} values are not less than u can be used. Summation over all i and j values satisfying that requirement is denoted by

$\sum^*_{i,j}$. Thus for non-negative values of u,

$$\sum^*_{i,j} \frac{\nu_i Q_{ij}}{Q}\left(1 - \frac{u}{Q_{ij}}\right) \qquad (2)$$

is the probability that the time required to print at a given position exceeds u, given that some character will be printed there.

It is clear that, if printing time for an entire line containing W characters is not to exceed u, the printing time at every one of the W occupied print positions must not exceed u. Thus, the joint probability that $W = w$ and print time for the line does not exceed u is

$$C^M_w p^w q^{M-w}\left[1 - \sum^*_{i,j} \frac{\nu_i Q_{ij}}{Q}\left(1 - \frac{u}{Q_{ij}}\right)\right]^w, \quad u > 0, \qquad (3)$$

and the case $u = 0$ is trivial.

The symbol U is adopted to represent the time to print a line, its cumulative probability distribution being denoted by $F(u)$. Thus, for positive values of u,

$$\begin{aligned} P(U \le u) &= F(u) \\ &= \sum_{w=0}^{M} C^M_w p^w q^{M-w}\left[1 - \sum^*_{i,j} \frac{\nu_i Q_{ij}}{Q} \times \left(1 - \frac{u}{Q_{ij}}\right)\right]^w \\ &= \left[1 - \sum^*_{i,j} p\frac{\nu_i Q_{ij}}{Q}\left(1 - \frac{u}{Q_{ij}}\right)\right]^M. \end{aligned} \qquad (4)$$

Evaluation is somewhat easier to manage if terms in the summation are collected into groups so that character copies represented in each group have identical Q_{ij} values. Thus, if there are k distinct Q_{ij} values, we can relabel them $Q_1, \cdots, Q_k$ with the agreement that $Q_1 > Q_2 > \cdots > Q_k$. (The special case, $k = 1$, is discussed later.)

This permits evaluation of $F(u)$ between 0 and Q_k, between Q_k and Q_{k-1}, etc. Specifically, (for $r = 1, \cdots, k$)

$$F(u) = \begin{cases} 1 & , \quad u \ge Q_1 \\ \left[1 - \sum_{m=1}^{r} p_m\left(1 - \frac{u}{Q_m}\right)\right]^M, & Q_{r+1} \le u < Q_r \\ 0 & , \quad \text{elsewhere}, \end{cases} \qquad (5)$$

in which

$$Q_{k+1} = 0$$

$$p_m = \sum_{i,j(m)} p\nu_i Q_{ij}/Q, \qquad m = 1, \cdots, k$$

and the summation indicated by $\sum_{i,j(m)}$ is taken over all copies (of all characters) for which the associated Q_{ij} values are equal to Q_m.

It is relevant to note that

$$\sum_{m=1}^{r+1} p_m\left(1 - \frac{Q_{r+1}}{Q_m}\right) = \sum_{m=1}^{r} p_m\left(1 - \frac{Q_{r+1}}{Q_m}\right), \qquad (6)$$

a relationship which is used to simplify results in subsequent paragraphs.

- *Minimum time T_0 specified*

It was pointed out in Part I that physical limitations may restrict skipping from one line to the next. With each line, a specified minimum time T_0 must elapse before the next line is begun. The effective print time in such printers cannot be less than T_0 no matter how quickly all characters on a line are printed. In that context, the effective time to print a line is designated by T and, hereafter, it is assumed that $0 \le T_0 < Q_1$.

Preceding developments permit the assertion that

$$P(T \le t) = \begin{cases} F(t), & t \ge T_0 \\ 0 \quad, & \text{otherwise}, \end{cases} \qquad (7)$$

where $F(t)$ as defined by Eq. (5) is a function of k and the Q_i values.

- *Elementary properties of T*

Some new parameters are introduced for convenience. Let

$$\gamma_r = 1 - \sum_{m=1}^{r} p_m,$$

$$\begin{aligned} \mu_r &= \sum_{m=1}^{r} \sum_{i,j(m)} [p\nu_i Q_{ij}/Q]/Q_{ij} \\ &= \sum_{m=1}^{r} \sum_{i=1}^{n} r_{i(m)} p\nu_i/Q, \end{aligned} \qquad (8)$$

in which $r_{i(m)}$ is the number of copies of character C_i having Q_{ij} values in excess of Q_{m+1}. Then

$$P(T \le t) = \begin{cases} 1 & , \quad t \ge Q_1 \\ (t\mu_r + \gamma_r)^M, & Q_{r+1} \le t < Q_r \text{ and } t \ge T_0 \\ 0 & , \quad \text{otherwise}. \end{cases} \qquad (9)$$

Derivatives of $P(T \le t)$ with respect to t are discontinuous at one or more points in the interval $t \ge 0$. Taking due care of that situation, the function

$$f(t) = \begin{cases} F(T_0) & , \quad t = T_0 \\ M\mu_r(t\mu_r + \gamma_r)^{M-1}, & Q_{r+1} \le t < Q_r \text{ and } t > T_0 \\ 0 & , \quad \text{elsewhere}, \end{cases} \qquad (10)$$

for $r = 1, \cdots, k$ is adopted in the role of a probability density function for T (though use of the phrase "density function" may stimulate minor objection on the part of some readers).

Moments of T can be expressed with the help of functions defined by

$$h_j(s) = \int_{Q_{j+1}}^{Q_j} t^s f(t)\,dt = \frac{M}{M+s}\left[Q_j F_j - Q_{j+1} F_{j+1} - \frac{s\gamma_j}{M\mu_j} h_j(s-1)\right] \quad (11)$$

for $Q_{j+1} > T_0$ and $s \geq 1$, the symbol F_j indicating $P(T \leq Q_j)$. In simpler notation, $F_j = F(Q_j)$ and the s^{th} moment of T is written

$$E(T^s) = T_0^s F + \int_{T_0}^{Q_{K-1}} t^s f(t)\,dt + \sum_{j=1}^{K-2} h_j(s)\,, \quad (12)$$

in which $F = F(T_0)$, K indicates the subscript on the largest Q_m value which does not exceed T_0, and the summation indicated by $\sum_{j=1}^{K-2}$ vanishes in the case $K \leq 2$.

The expected value of T (i.e., the expected effective time to print a line) is $E(T)$ while the variance of T is $E(T^2) - [E(T)]^2$. Specifically,

$$E(T) = \frac{T_0 F}{M+1} + \frac{M}{M+1} \times \left(Q_1 - \frac{1}{M}\left[\frac{\gamma_{K-1}}{\mu_{K-1}}\{F_{K-1} - F\} + \sum_{j=1}^{K-2} \frac{\gamma_j}{\mu_j}\{F_j - F_{j+1}\}\right]\right),$$

$$E(T^2) = \frac{T_0^2 F}{M+2} + \frac{M}{M+2} \times \left(Q_1^2 - \frac{2}{M+1}\left[\frac{\gamma_{K-1}}{\mu_{K-1}}\{Q_{K-1}F_{K-1} - T_0 F - \frac{1}{M}\frac{\gamma_{K-1}}{\mu_{K-1}}(F_{K-1} - F)\} + \sum_{j=1}^{K-2} \frac{\gamma_j}{\mu_j}\{Q_j F_j - Q_{j+1}F_{j+1} - \frac{1}{M}\frac{\gamma_j}{\mu_j}(F_j - F_{j+1})\}\right]\right), \quad (13)$$

with the agreement that sums indicated by $\sum_{j=1}^{K-2}$ vanish if $K \leq 2$. (Relationship (6) is particularly useful in arriving at the particular form in which Eq. (13) is given.)

• *A special case:* $k = 1$

"Standard" print chains are designed so that characters are repeated with equal frequency on the chain, and copies of each character are equally spaced along the chain. This case implies $k = 1$ and, if $q_1 = 1 - p_1$,

$$E(T) = \frac{T_0 F}{M+1} + Q_1\left[1 - \frac{1 - q_1 F}{(M+1)p_1}\right],$$

$$E(T^2) = \frac{2T_0^2 F}{M+2} + \frac{M}{M+2} \times \left[Q_1^2 - \frac{2Q_1 q_1}{Mp_1}\left\{E(T) - \frac{T_0 F}{M+1}\right\}\right]. \quad (14)$$

• *Printing groups of lines*

Return now to the more general case, i.e., to $k \geq 1$, and the corresponding formulas. All moments of T are finite so that the Central Limit Theorem of statistics applies when a large group of R independent lines is printed.

In that context, suppose that incrementing the paper from line to line is ignored. Let T^* be the effective time to print R independent lines and let $\xi = E(T)$ in the general case. Then the asymptotic distribution of the variable

$$\frac{T^* - R\xi}{\sigma_T \sqrt{R}}, \quad (15)$$

(where $\sigma_T{}^2$ is the variance of T) is normal (Gaussian) with zero mean and unit standard deviation. Hence, when R is large enough, the variable T^* is approximately normally distributed with $R\xi$ as its expected value and $\sigma_T \sqrt{R}$ as its standard deviation.

• *A limiting case*

Approximation of $E(T)$ and $E(T^2)$ is desirable when M is large with respect to p. Toward that end, consider the case in which M tends to infinity and p tends to zero in such a fashion that the product Mp tends to a constant λ. An arrow ($\rightarrow$) is used to indicate passing to the limit in that fashion and the notation $\lambda_i = \lambda\nu_i$ is adopted. Then, referring to Eq. (8) and the definitions which follow Eq. (5),

$$\gamma_r \rightarrow 1$$
$$\mu_r \rightarrow 0$$
$$M\mu_r \rightarrow \sum_{m=1}^{r} \sum_{i=1}^{n} r_{\,i(m)}\lambda_i/Q$$
$$(t\mu_r + \gamma_r)^M \rightarrow \exp(-\alpha_r + t\beta_r)\,, \quad (16)$$

in which

$$\alpha_r = \sum_{m=1}^{r} \sum_{i,j(m)} \lambda_i Q_{ij}/Q$$

and β_r is the expression to which $M\mu_r$ tends, as is also shown in Eq. (16).

Corresponding limiting forms of $F(t)$, $f(t)$, $E(T)$ and $E(T^2)$ are useful and are obtained easily by inspection of their respective definitions. For example, using the notation $\tilde{F}$ and $\tilde{F}_j$ to indicate the limiting values of F and F_j respectively, reference to Eq. (13) indicates

$$E(T) \rightarrow Q_1 - \left[\frac{\tilde{F}_{K-1} - \tilde{F}}{\beta_{K-1}} + \sum_{j=1}^{K-2} \frac{\tilde{F}_j - \tilde{F}_{j+1}}{\beta_j}\right],$$

$$E(T^2) \rightarrow Q_1^2 - 2\left[\frac{Q_{K-1}\tilde{F}_{K-1} - T_0\tilde{F}}{\beta_{K-1}} - \frac{\tilde{F}_{K-1} - \tilde{F}}{\beta_{K-1}^2} + \sum_{j=1}^{K-2}\left\{\frac{Q_j\tilde{F}_j - Q_{j+1}\tilde{F}_{j+1}}{\beta_j} - \frac{\tilde{F}_j - \tilde{F}_{j+1}}{\beta_j^2}\right\}\right]. \quad (17)$$

- *Separation of character copies*

It is natural to ask what is the best way to arrange copies for any character on the chain. In the present context, the best arrangement must achieve a minimum average print time per line.

All examples which the authors have examined indicate that equal spacing of character copies gives the best arrangement. That is to say, if there are r_i copies of character C_i, the corresponding Q_{ij} values should be equal so that $Q_{ij} = Q/r_i$ for each copy.

Though no rigorous mathematical support has been devised, the authors adopt the equal spacing notion throughout Part III. The matter of selecting a best chain is approached in two distinct phases. First, it is assumed that equal spacing of copies is always possible for each character regardless of the r_i values involved. A best chain is found under that assumption. The second phase consists of designing a practical character copy arrangement which approximates the best "equal spacing" configuration as closely as possible.

- *Summary of equal spacing results*

It is easy to determine the effects of assumed equal copy spacing when results given in preceding sections are considered. Suppose that groups of characters are formed according to the number of copies found on the chain, i.e., all characters which appear on the chain exactly r_m times belong to the same group. Suppose further that a total of k such groups can be formed among the characters $C_1, \cdots,$ C_n, and with no loss of generality, that $r_1 < r_2 < \cdots < r_k$ denote the k distinct values found among the numbers $r_1,$ $\cdots, r_n$. The symbol $\sum_m'$ is adopted ($m < k$) to denote summation over all indicated values for characters which appear on the chain r_m times (i.e., for all characters in the group associated with r_m).

Then the symbols

$$p_m' = \sum_m' p\nu_i\,, \qquad \lambda_m' = \sum_m' \lambda_i\,,$$
$$\gamma_r' = 1 - \sum_{m=1}^{r} p_m'\,, \qquad \alpha_r' = \sum_{m=1}^{r} \lambda_m'\,, \tag{18}$$
$$\mu_r = \sum_{m=1}^{r} p_m/Q_m\,, \qquad \beta_r' = \sum_{m=1}^{r} \lambda_m'/Q_m\,,$$

permit simplification of previous results when applied to the equal spacing case.

One merely substitutes p_m' for p_m, γ_r' for γ_r, etc., in the appropriate formula from the previous paragraphs, Eqs. (9), (10), (13), (16), and (17) being of special interest. For example, in the special case of equal spacing

$$P(T \le t) = \tilde{F}(t)$$
$$\rightarrow \begin{cases} 1 & , \quad t \ge Q_1 \\ \exp(-\alpha_r' + t\beta_r')\,, & Q_{r+1} \le t \le Q_r \text{ and } t \ge T_0 \\ 0 & , \quad \text{elsewhere}\,, \end{cases} \tag{19}$$

and

$$E(T) \rightarrow Q_1 - \left[\frac{\tilde{F}_{K-1} - \tilde{F}}{\beta_{K-1}'} + \sum_{j=1}^{K-2} \frac{\tilde{F}_j - \tilde{F}_{j+1}}{\beta_j'}\right], \tag{20}$$

in which α_j' and β_j' are used in determining $\tilde{F}$ and $\tilde{F}_j$ for all values of j.

Part III. Iterative optimization procedure

In the previous section an expression for the expected time to print a line, $E(T)$, (hereafter denoted by E) was derived in terms of the character usage, the number of copies of each character on the chain, the number of character positions on the chain, and the minimum line-print time. This expression will now be used in an iterative procedure for determining how many copies of each character should be placed on the chain for the printer to operate at or near maximum speed.

It will be assumed that the character usage statistics and printer operating characteristics are fixed, thus making E a function of the n variables $r_1, r_2, \cdots, r_n$ (where r_i is the number of copies of character C_i). The expression given for E in Eq. (20) is based on the assumption that if the character C_i is on the chain r_i times, its copies are equally spaced around the chain.

The basic problem then is to determine positive integer values for the n variables $r_1, r_2, \cdots, r_n$, such that:

(i) $\sum_{i=1}^{n} r_i = Q$, and

(ii) E is a minimum.

Since E is a complicated function of the n integer valued variables $r_1, r_2, \cdots, r_n$, the function will be minimized by selecting some reasonable initial set of values for the n variables and then systematically changing these values until a local minimum is determined for E. Experience has indicated that in most cases only one local minimum can be found. (In those few cases where more than one local minimum was found, the corresponding E values differed by a negligible amount.)

- *Obtaining a local minimum for E*

The smallest possible modification of the n variables (subject to the condition that the sum of their integer values must be Q) is to add "1" to some variable r_i, and subtract "1" from some other variable r_j. It will be assumed that a local minimum has been reached if no modification of this type will further reduce the value of E. (In terms of the printer, this is equivalent to saying that we will have obtained the best print chain definition if it is impossible to make the printer run any faster by adding a copy of some character C_i at the expense of removing a copy of some other character C_j.)

The basic step in the iterative algorithm will be to select a pair of variables (r_x, r_y) that will result in the largest possible reduction in E when they are modified by adding "1" to r_x and subtracting "1" from r_y. The algorithm will iterate on this step and terminate when no (r_x, r_y) pair can be found that will further reduce the value of E.

The key problem in each step is to identify the (r_x, r_y) pair that will yield the largest reduction in E. This could be accomplished by evaluating E for all possible (r_i, r_j) pairs. For n variables, this involves a total of $n(n-1)$ evaluations of E. The number of evaluations of E can be greatly reduced by making use of the following practical considerations.

If some character C_i is printed more frequently than some other character C_j, clearly, for maximum printing speed, character C_i should have at least as many copies on the chain as does character C_j. Thus, the following relationship must be true for maximum printing speed or minimum E:

$$r_i \geq r_j \quad \text{if} \quad e_i > e_j \,.$$

This means that if two or more characters have the same r_i value, we only need to consider adding a copy of the character with the largest value of e_i and we only need to consider substracting a copy of the character with the smallest value of e_i.

It will be convenient to partition the characters into k disjoint groups such that C_i and C_j belong to the same group if and only if $r_i = r_j$. At each step in the algorithm it will only be necessary to consider the most used and the least used (largest and smallest e_i value) member of each group as possible candidates for r_x and r_y respectively. This will reduce the number of evaluations of E from $n(n-1)$ to $k(k-1)$.

A further reduction in the number of evaluations of E can be achieved by selecting r_x and r_y independently. We can determine r_x by evaluating the changes in E resulting from separately adding "1" to the r_i value of the most used (largest e_i) character in each group. The r_i variable giving the largest reduction in E will be selected for r_x. Similarly, r_y can be selected by separately subtracting "1" from the r_i value of the least used character in each group and selecting for r_y the one giving the smallest increase in E. This technique will reduce the number of evaluations of E to $2k + 1$ for each step in the iteration. If the (r_x, r_y) pair selected in this manner does not result in a decrease in the value of E, it is then worthwhile evaluating all the $k(k-1)$ possible candidates for (r_x, r_y) to make sure that no other combination will result in a decrease in the value of E. This also guarantees that a local minimum has been obtained.

The following is a brief summary of the previously discussed algorithm for obtaining a local minimum for E:

1. Determine an initial set of positive integer values for the r_i variables such that $\sum_{i=1}^{n} r_i = Q$.
2. Partition the characters into k disjoint groups such that C_i and C_j will belong to the same group if $r_i = r_j$.
3. Select the (r_x, r_y) pair in the following way:
   a. For each of the k groups determine the change in E resulting from adding 1 to the r_i value of the character with the largest e_i in that group. Select for r_x that r_i variable corresponding to the largest decrease in E.
   b. For each of the k groups whose r_i values are greater than 1 evaluate the change in E resulting from subtracting "1" from the r_i value of the character with the smallest e_i in that group. Select for r_y that r_i variable corresponding to the smallest increase in E. (In the unlikely event that r_x and r_y turn out to be the same variable, select for r_y the r_i variable corresponding to the second smallest increase in E.)
4. Modify the r_i values by adding 1 to r_x and subtracting 1 from r_y and evaluate the change in E.
   a. If the new value of E is less than the previous value, repeat the procedure beginning at step 2.
   b. If the new value of E is not less than the previous value, go to step 5.
5. With the r_i values prior to step 4 evaluate the change in E resulting from adding and subtracting 1 to r_x and r_y, respectively, for all possible combinations of r_x and r_y, where C_x is the most used member of some group and C_y is the least used member of some group whose r_i values are greater than 1. If an (r_x, r_y) pair is found that further reduces the value of E, update the r_i values accordingly and go to step 2. If no (r_x, r_y) pair is found that reduces the value of E, a local minimum has been obtained for E.

Part IV. Validation tests

In this section we present results which indicate that the two-fold purpose mentioned in the Introduction can be achieved using the methods of this paper; i.e., given character usage statistics we can (1) reliably estimate printing speed for a given character chain arrangement and (2) improve the arrangement so as to increase the average printing speed.

We shall give three examples of evaluation and "optimization." The first two are concerned with usage statistics from two separate printing applications. The third illustrates what can be done with usage statistics from a number of different printing applications when "optimization" is based on the composite statistics.

The usage statistics for Example 1 come from printing samples in the area of cost accounting, while those of Example 2 come from inventories. Usage statistics were gathered from 10,000 lines of output for each of the two jobs and an "optimum" chain configuration was determined for each. The tests were run on an IBM 1403 Model 2 Printer with the Universal Character Set feature and were accomplished by loading the IBM 2821 Control Unit Buffer with the 240 characters corresponding to each of the various chain configurations used. A 5000-line sample from each

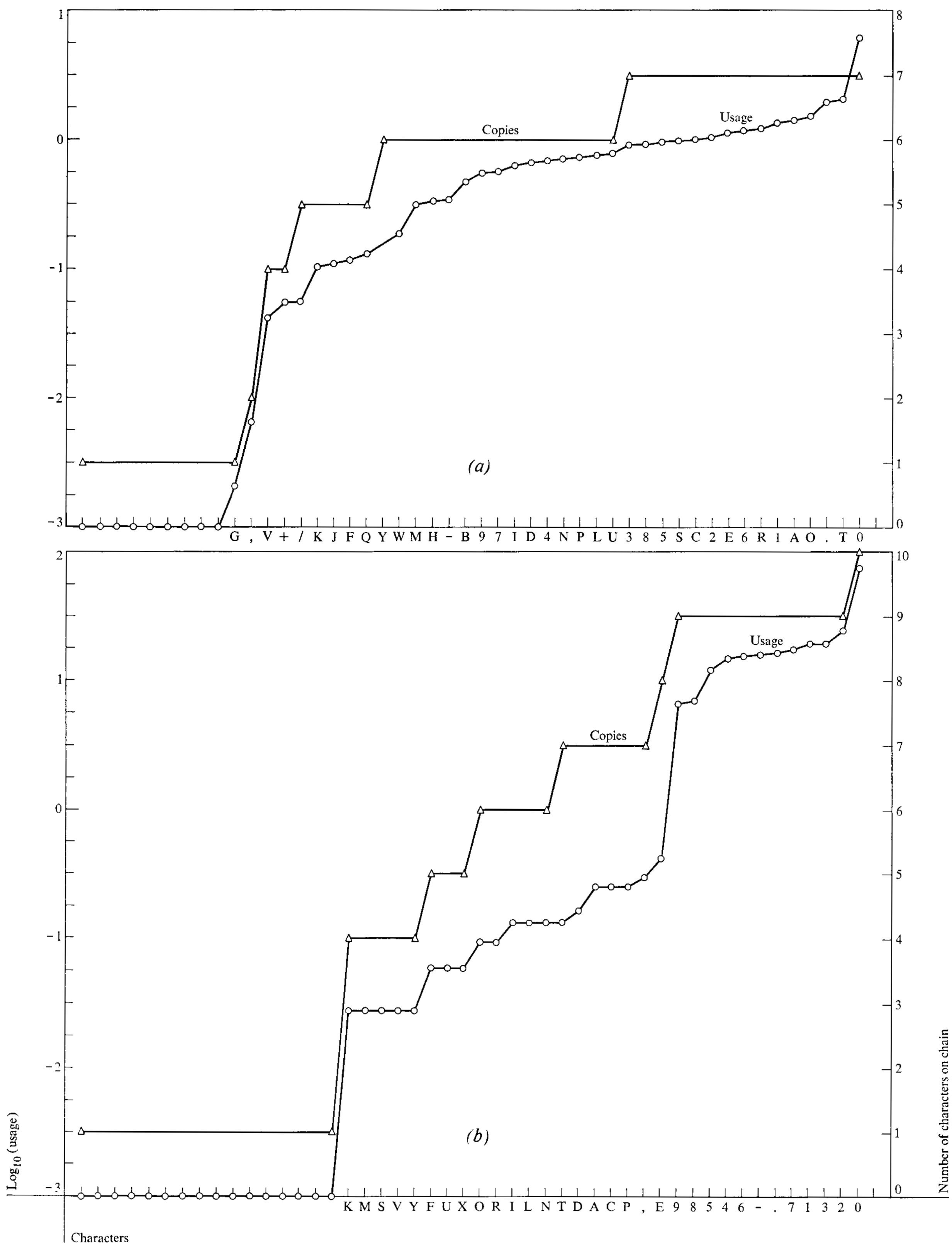
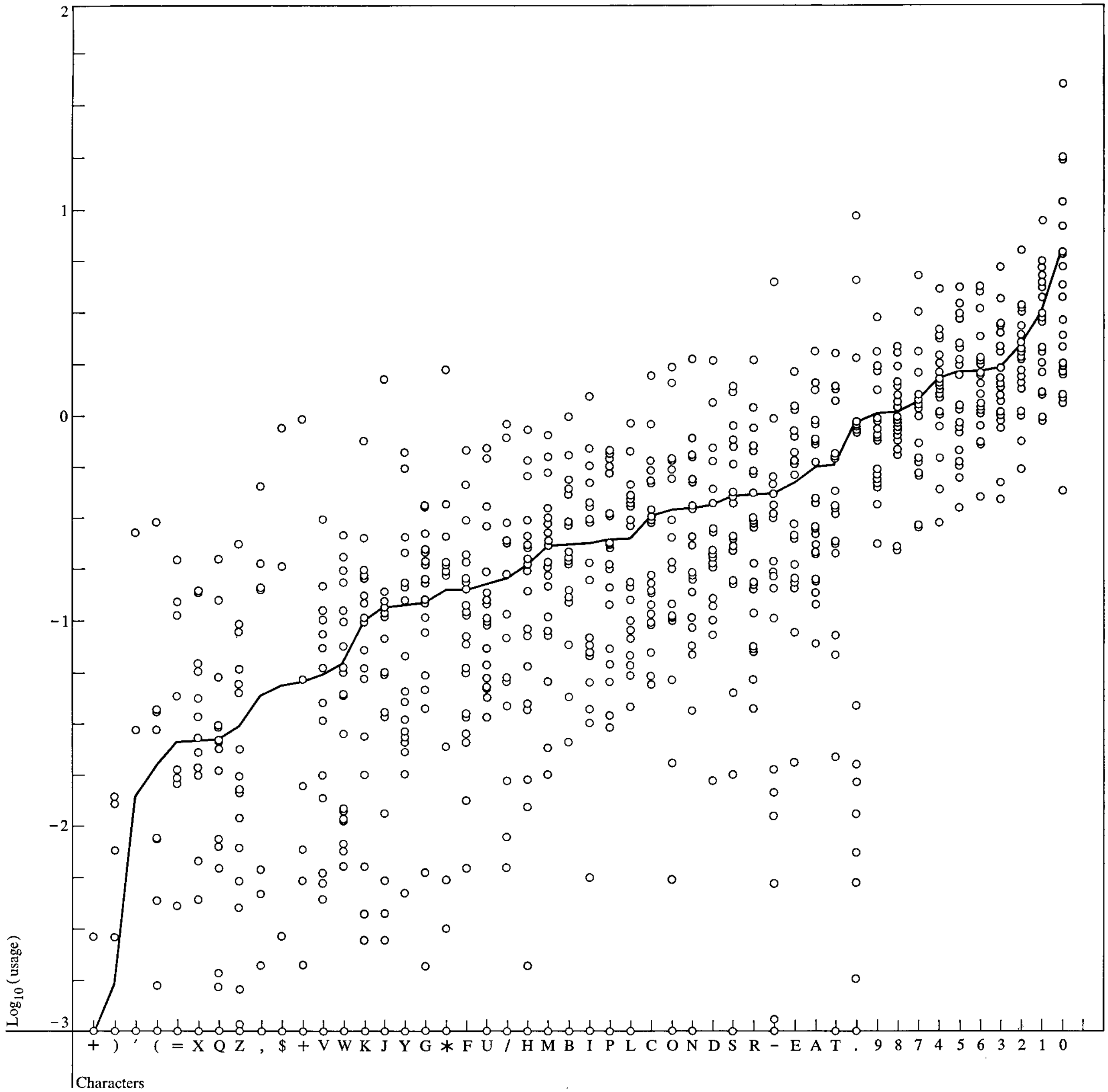


Figure 4 Number of character copies and $\log_{10}$ of average use per line of each character on optimum chain. Characters used less than 10^{-3} times per line are not identified and their usage statistics are shown as 10^{-3}. (a) Example 1: cost accounting application; (b) Example 2: inventory application.

Table 1 Calculated and measured printing speeds for test examples

	Example 1—Cost accounting			*Example 2—Inventory*		
	Standard chain	*Optimum chain*	*% Improvement*	*Standard chain*	*Optimum chain*	*% Improvement*
Measured	605 LPM	694 LPM	14.7	584 LPM	741 LPM	26.9
Calculated	605 LPM	702 LPM	16.1	597 LPM	749 LPM	25.5
Error	0%	1.15%		2.22%	1.08%	

Figure 5 Log_{10} of usage statistics for each character on a chain optimized for a composite of 22 different applications. The solid line shows the average for all applications.



job was printed twice, once with the standard chain (each character on chain five times) and once with the corresponding "optimum chain." It should be noted that the contents of the buffer did not match the actual characters on the physical chain in the case of the "optimum chains." In order to facilitate comparisons between theory and experiment, single spacing was forced; printing was done in batches of 1,000 lines and measured with a stop watch. Table 1 presents the results of the tests. The standard times of 1.665 msec/scan and 21.7 msec/carriage advance were used in determining the calculated speeds.

The logarithms of the character usage statistics of Examples 1 and 2 are shown plotted in Figs. 4a and b along with the number of times each character occurred on the corresponding "optimum chain." The characters are ordered according to their respective usage. Note that at least one copy of each character in the character set occurs on these chains even though it is not used. Also note the wide range in character usage, especially in Example 2.

The results of the foregoing tests indicate that significant increases in printing speed may be obtained provided the usage statistics are indicative of the printing demands.

Often, however, printers are called upon to print output from a wide variety of applications, so that the character usage statistics for a given job may vary drastically from, say, the average character usage which the printer must handle.

In order to examine this effect, usage statistics from some 22 different types of printing jobs (Examples 1 and 2 included) were equally weighted and combined for Example 3. The average number of characters printed per line varied from about 13 to about 70. The logarithms of the usages for each character are plotted as points in Fig. 5, arranged in order of the composite usage. The composite usage is shown as a continuous curve in order to emphasize the scatter about the average. A chain was optimized according to the composite statistics and the printing speed of each job was

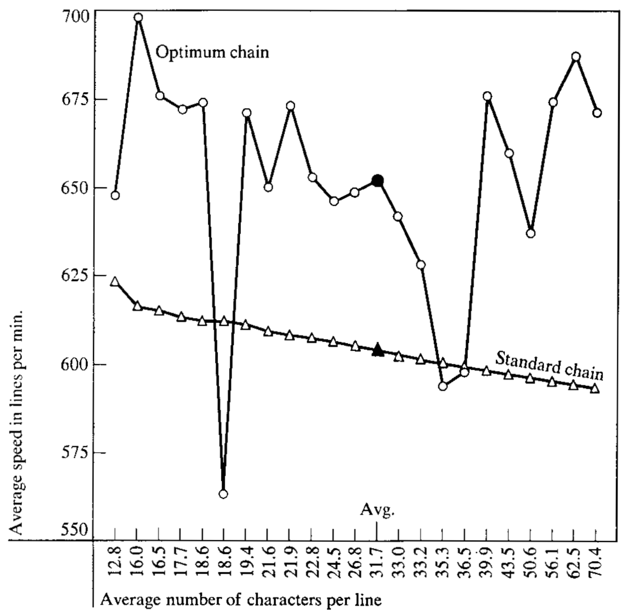


Figure 6 Average printing speed for each of 22 applications using standard chain and chain optimized with composite usage statistics.

then individually computed assuming the same type of printer as in the first two examples. The results are plotted in Fig. 6 and show that moderate increases in printing speeds can be achieved even when based on a broad range of applications.

Acknowledgments

We wish to acknowledge the support and encouragement of J. E. Brophy, T. F. Nealon, and R. E. Norwood and the assistance of J. F. Dubil and J. T. Sermarini.

Received August 17, 1966.
Revised September 13, 1967.