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Fourier Analysis of the Motion of a Hydraulically Controlled Piston

Abstract: The problem is considered of the motion of a free piston in a finite pipe filled with a compressible liquid and subject to a step in pressure introduced from the two ends. The treatment is in one dimension using the linearized wave equation for the density disturbance and the linearized boundary conditions. A generalized Fourier series expansion leads to the solution of the problem. The mathematical analysis is complicated first by the presence of the interior boundary conditions which lead to a system of discontinuous eigenfunctions, and second, by the step pressure input which results in reflected discontinuities. By studying the properties of the eigenfunctions from a variational characterization, the formal expansions used are established rigorously. The motion of the piston is determined as a function of the input parameters, and the maximum piston excursion and the associated time are tabulated numerically over the ranges of interest of these parameters.

1. Introduction

Pressure pulses in hydraulic lines can be used to drive or position pistons with high speed and precision. In a variety of applications, such mechanisms are superior to direct linkage drives. The usual hydraulic analysis of the motion of the piston ignores wave motion in the fluid, relating pressures directly to volume changes. The purpose of the present paper is to give a complete linear treatment of the motion of a coupled fluid-piston-fluid-system.

The system to be considered consists of a free piston in a finite pipe filled with hydraulic oil, with pressures introduced from the two ends. The oil will be treated as a compressible, inviscid liquid and the problem will be restricted to one dimension, thereby neglecting the influence of the wall of the tube. The disturbances will be assumed to be small so that linear equations for the wave motion and the boundary conditions result.

The solution of the linearized problem is accomplished by a generalized Fourier series expansion in which each term is a particular solution in product form of the reduced homogeneous problem.

While the approach is classical, there are two aspects of the problem which lift it above the ordinary and complicate the analysis. The first is due to the presence of the piston in the interior of the interval, leading to two interior "boundary conditions" in addition to those at the ends of the tube. This leads to a system of discontinuous eigenfunctions, the discontinuity resulting from the pressure discontinuity across the piston. In Appendix A, the properties of these eigenfunctions are discussed, in particular the orthogonality and completeness relations upon which the series expansions are based. These properties are derived from a variational characterization of the eigenfunctions and eigenvalues.

The second inherent complication lies in the fact that the pressure input at the ends of the tube introduces discontinuous waves at the initial time. Assuming small disturbances, these waves travel at sonic speed toward the piston, are reflected, and thereafter continue to bounce back and forth between the piston and the ends of the tube. Because of these discontinuities, the problem admits only of a so-called "weak solution." One cannot, therefore, directly establish that the formal series expansion for the density variation is actually the solution of the wave equation subject to the various boundary and initial conditions. Appendix B, however, establishes the validity of the expansion for the time integral of this function.

The final step in the solution is to determine the motion of the piston from the pressure variation on its two sides. Since the series obtained is rapidly convergent, good numerical results may be obtained.

In Section 7 the numerical calculations are first carried through explicitly for a special case where the eigenvalues and the Fourier coefficients may be found simply from asymptotic formulas. This is followed by the results, shown in Figs. 5-8, of a complete IBM 704 study determining maximum piston excursion and corresponding time required over a range of the basic nondimensional physical parameters specifying mass and geometry.

2. Equations of motion of the fluid

We consider the flow of a compressible fluid in one dimension, assuming no viscosity. The continuity equation expressing conservation of mass states that

$$\frac{\partial(\rho u)}{\partial x} = -\frac{\partial \rho}{\partial t}, \tag{2.1}$$

where x denotes the space variable, t the time and both ρ, the density (mass per unit volume), and u, the fluid velocity in the x direction, are functions of x and t. A second equation is a statement of Newton's law of motion, or conservation of linear momentum, given by

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x}, \tag{2.2}$$

where p is the pressure (force per unit area), a function of x and t and D/Dt stands for the Eulerian or material derivative

$$\frac{Du}{Dt} \equiv \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}. \tag{2.3}$$

A third equation is given by a pressure-density relation of the form[1]

$$p = p(\rho). \tag{2.4}$$

Letting ρ_0 denote the constant undisturbed density of the fluid at atmospheric pressure and ambient temperature, we write the total density $\rho(x, t)$ as

$$\rho = \rho_0 + \rho', \tag{2.5}$$

where $\rho'(x, t)$ is assumed to be small compared to ρ_0. Substituting (2.5) into (2.1) and (2.2) and discarding products of u and ρ' and their derivatives as being of higher order gives the equations

$$\rho_0 \frac{\partial u}{\partial x} + \frac{\partial \rho'}{\partial t} = 0 \tag{2.6}$$

and

$$\rho_0 \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0. \tag{2.7}$$

From (2.4),

$$\frac{\partial p}{\partial x} = \frac{dp}{d\rho}\frac{\partial \rho}{\partial x}$$

or to the first order,

$$\frac{\partial p}{\partial x} = a_0^2 \frac{\partial \rho'}{\partial x}, \tag{2.8}$$

where

$$a_0^2 = \left(\frac{dp}{d\rho}\right)\bigg|_{\rho=\rho_0}. \tag{2.9}$$

From (2.7) and (2.8)

$$\rho_0 \frac{\partial u}{\partial t} + a_0^2 \frac{\partial \rho'}{\partial x} = 0. \tag{2.10}$$

Eliminating ρ' first and then u from the pair of equations (2.6) and (2.10) gives the second-order equations

$$\frac{\partial^2 \rho'}{\partial x^2} = \frac{1}{a_0^2}\frac{\partial^2 \rho'}{\partial t^2} \tag{2.11}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a_0^2}\frac{\partial^2 u}{\partial t^2} \tag{2.12}$$

for the functions $\rho'(x, t)$ and $u(x, t)$. These are the well-known linear wave equations of acoustics, and the solution of either equation can be expressed as the sum of a function of $(x - a_0 t)$ and a function of $(x + a_0 t)$; these functions represent advancing and receding waves of velocity a_0. Thus the constant a_0 appears in this theory as the velocity of sound, i.e., small disturbances, in the fluid at density ρ_0.

3. Statement of the problem

We consider one-dimensional motion of a fluid-piston-fluid system as shown in Fig. 1. Distance x is measured from the left end of the tube. The piston, of mass M per unit cross-sectional area of the tube, is assumed at $t=0$ to be at rest at a distance l_1 from the left end of the tube and l_2 from the right end. (Since the piston is treated as a rigid body, the actual length of the piston does not enter. For convenience we shall assign it zero length and locate it in its entirety by a single coordinate.) Also at the initial time the fluid is assumed to be at rest throughout the tube at the undisturbed density ρ_0. At the boundaries of the tube, overpressures are assumed to be introduced at the initial instant and then maintained. The problem is to determine the ensuing motion of the piston.

The conditions so far described may be expressed in terms of the density variation $\rho'(x, t)$ as follows. For convenience we denote I_1 the interval $0 < x < l_1$ and by I_2 the interval $l_1 < x < l_1 + l_2$.

• *Initial conditions for x in I_1 and I_2*

$$\rho'(x, 0) = \rho_t'(x, 0) = 0; \tag{3.1}$$

• *Boundary conditions at tube ends for $t > 0$*

$$\rho'(0, t) = \rho_1', \ \rho'(l_1 + l_2, t) = \rho_2', \tag{3.2}$$

where the subscript t denotes partial differentiation with respect to time and ρ_1' and ρ_2' are prescribed positive constants.

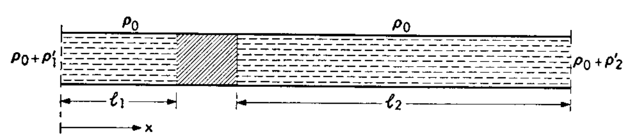

Figure 1 **Configuration of the piston and tube at initial time.**

Still to be formulated is the condition at the piston-fluid interfaces. Let the variable $\xi=\xi(t)$ denote the position of the piston at time t, measured from the initial position at $x=l_1$. Then

$$\xi(0)=\dot{\xi}(0)=0\,, \tag{3.3}$$

where the dot denotes total differentiation with respect to t. The density of the fluid adjacent to the piston on the left at time t is $\rho_0+\rho_-'$, where ρ_-' denotes the limit of $\rho'(x,t)$ as x approaches $l_1+\xi(t)$ from the left. Similarly, the density on the right-hand side is denoted by $\rho_0+\rho_+'$. Using the linear terms in the Taylor expansion of the pressure $p=p(\rho)$ as a function of density, we can therefore find the resultant force on the piston and write the equation of motion as

$$M\ddot{\xi}=p_--p_+=a_0^2(\rho_-'-\rho_+')\,, \tag{3.4}$$

taking into account (2.9).

Since the right side of (3.4) contains in the arguments the unknown position $\xi(t)$ of the piston, this equation leads to an inherently nonlinear condition. Consistent with the assumption of small disturbances and a linear theory, however, we may assume $\xi(t)$ small and evaluate ρ_-' and ρ_+' at the original interface location $\xi=0$. Thus in the following we denote

$$\rho_\pm'=\lim_{x\to l_1\pm 0}\rho'(x,t)\,.$$

By continuity of the fluid, the piston velocity and the fluid velocity immediately adjacent to the piston must be the same, so that to a linear approximation

$$\lim_{x\to l_1-0}u(x,t)=\lim_{x\to l_1+0}u(x,t)=\dot{\xi}(t)\,.$$

Upon differentiation we have

$$\frac{\partial u_-}{\partial t}=\frac{\partial u_+}{\partial t}=\ddot{\xi}\,, \tag{3.5}$$

where the subscripts $-$ and $+$ denote left- and right-hand limits at $x=l_1$ as before. From Eq. (2.10) we have

$$\frac{\partial u_\pm}{\partial t}=-\frac{a_0^2}{\rho_0}\frac{\partial\rho_\pm'}{\partial x}\,. \tag{3.6}$$

Combining (3.4), (3.5) and (3.6)

$$\frac{\partial\rho_+'}{\partial x}=\frac{\partial\rho_-'}{\partial x}=\frac{\rho_0}{M}\,[\rho_+'-\rho_-']\,, \tag{3.7}$$

constituting two additional boundary conditions to be satisfied for all $t>0$ at $x=l_1$. These conditions state that the left- and right-hand derivatives of $\rho'(x,t)$ at $x=l_1$ should be equal and have the value (ρ_0/M) times the jump in $\rho'(x,t)$.

The problem stated in terms of the fluid is to determine a solution $\rho'(x,t)$ of (2.11) satisfying the initial conditions (3.1) and the boundary conditions (3.2) and (3.7). Because of the initial density discontinuities at the ends of the tube, the solution $\rho'(x,t)$ will be expected to have discontinuities along the characteristics of the wave equation issuing from these points and the subsequent reflections of these characteristics from the boundaries. Consequently we require that $\rho'(x,t)$ satisfy the wave equation (2.11) for x in I_1 and I_2 with $t>0$ except at most along certain lines in the $x-t$ plane across which $\rho'(x,t)$ may be discontinuous. (These discontinuities are subject to conditions of conservation, but it will not be necessary to formulate these explicitly for this discussion.) From the function $\rho'(x,t)$, the displacement $\xi(t)$ of the piston can be obtained from (3.4) by two time integrations subject to (3.3).

4. Reduction of the problem

Before proceeding, we shall introduce nondimensional variables into our preceding equations and restate the problem in terms of these variables. Let

$$\bar{x}=\frac{\pi x}{l_1+l_2}\,,\qquad \tau=\frac{a_0\pi t}{l_1+l_2} \tag{4.1}$$

and

$$\bar{\rho}'(\bar{x},\tau)=\rho'(x,t)/\rho_0\,. \tag{4.2}$$

Denoting the reduced total density by $\bar{\rho}=\rho/\rho_0$ we have

$$\bar{\rho}=1+\bar{\rho}'\,. \tag{4.3}$$

The wave equation becomes

$$\frac{\partial^2\bar{\rho}'}{\partial\tau^2}=\frac{\partial^2\bar{\rho}'}{\partial\bar{x}^2} \tag{4.4}$$

over the ranges $0<\bar{x}<\theta\pi$, $\theta\pi<\bar{x}<\pi$, where

$$\theta=\frac{l_1}{l_1+l_2} \tag{4.5}$$

and $\bar{x}=\theta\pi$ is the initial position of the piston. Similarly as before we let $\bar{I}_1$ and $\bar{I}_2$ denote the intervals of $\bar{x}$ to the left and right of the piston respectively.

The initial conditions become

$$\bar{\rho}'(\bar{x},0)=\bar{\rho}_\tau'(\bar{x},0)=0 \tag{4.6}$$

for $\bar{x}$ in $\bar{I}_1$ and $\bar{I}_2$, and the boundary conditions at the ends of the tube are, for $\tau>0$,

$$\bar{\rho}'(0,\tau)=\bar{\rho}_1'\,,\quad \bar{\rho}'(\pi,\tau)=\bar{\rho}_2'\,, \tag{4.7}$$

where

$$\bar{\rho}_1'=\rho_1'/\rho_0\,,\quad \bar{\rho}_2'=\rho_2'/\rho_0\,. \tag{4.8}$$

At the piston, $\bar{x}=\theta\pi$, the condition (3.7) becomes

$$\frac{\partial\bar{\rho}_-'}{\partial\bar{x}}=\frac{\partial\bar{\rho}_+'}{\partial\bar{x}}=h[\bar{\rho}_+'-\bar{\rho}_-']\,, \tag{4.9}$$

where

$$h=\frac{\rho_0(l_1+l_2)}{M\pi} \tag{4.10}$$

and left- and right-hand limits refer to this value of $\bar{x}$.

We proceed now to reduce the problem to one with homogeneous boundary conditions replacing (4.7), by introducing a particular solution f of the wave equation which satisfies the given boundary conditions. We then express the desired solution $\bar{p}'$ as the difference between f and a function $r(\bar{x}, \tau)$ to be determined. It follows that r satisfies homogeneous boundary conditions, but since f turns out to satisfy one but not both initial conditions, r will be required to satisfy one nonhomogeneous initial condition in order to compensate for this. Consider the function

$$f(\bar{x}) \equiv \begin{cases} \dfrac{\bar{h}(\bar{p}_2'-\bar{p}_1')\bar{x}}{\pi\bar{h}+1} + \bar{p}_1', & \bar{x} \in \bar{I}_1, \\ \dfrac{\bar{h}(\bar{p}_2'-\bar{p}_1')\bar{x}}{\pi\bar{h}+1} + \dfrac{\bar{p}_2'+\pi\bar{h}\bar{p}_1'}{\pi\bar{h}+1}, & \bar{x} \in \bar{I}_2. \end{cases} \quad (4.11)$$

This function consists of two linear segments in $\bar{x}$ and does not involve τ, and thus trivially satisfies the wave equation. The slopes and intercepts have been chosen to satisfy the four boundary conditions (4.7) and (4.9) written for f, as may be checked directly. The graph of $f(\bar{x})$ is shown in Fig. 2. If we now define $r(\bar{x}, \tau)$ by the equation

$$\bar{p}' = f - r \quad (4.12)$$

it follows by the linear character of the problem that r must satisfy the following conditions:

$$\frac{\partial^2 r}{\partial \bar{x}^2} = \frac{\partial^2 r}{\partial \tau^2}, \; \bar{x} \in \bar{I}_1 \cup \bar{I}_2, \quad \tau > 0 \quad (4.13)$$

(excepting discontinuity lines of the function $r(\bar{x}, \tau)$)

$$r(0, \tau) = r(\pi, \tau) = 0, \quad \tau > 0, \quad (4.14)$$

$$\left.\begin{aligned} r(\bar{x}, 0) &= f(\bar{x}) \\ r_\tau(\bar{x}, 0) &= 0 \end{aligned}\right\} \bar{x} \in \bar{I}_1 \cup \bar{I}_2 \quad (4.15)$$

$$\frac{\partial r_-}{\partial \bar{x}} = \frac{\partial r_+}{\partial \bar{x}} = \bar{h}[r_+ - r_-]. \quad (4.16)$$

5. The formal Fourier solution

We begin by seeking particular solutions of the wave equation (4.13) subject to the boundary conditions (4.14) and (4.16) in the form of a product of a function of $\bar{x}$ by a function of τ, say $X(\bar{x})T(\tau)$. Substituting this expression in (4.13) we find in the usual way that

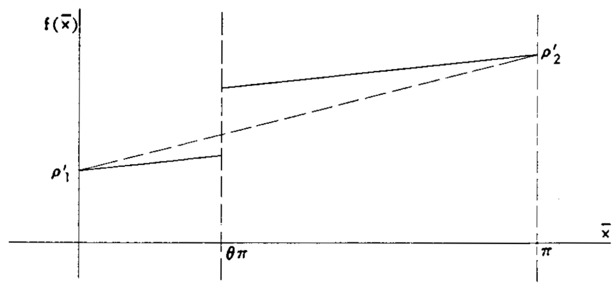

Figure 2 **Graph of $f(\bar{x})$ for $\rho_2' > \rho_1'$.**

$$X'' + \lambda^2 X = 0, \; \bar{x} \in \bar{I}_1 \cup \bar{I}_2 \quad (5.1)$$

$$T'' + \lambda^2 T = 0, \; \tau > 0, \quad (5.2)$$

where λ^2 is a constant.

Ordinarily, (5.1) is subject merely to two end conditions leading to a specification of functions X and corresponding λ values. In the present problem there are four conditions to be satisfied in (4.14) and (4.16). This is handled by allowing $X(\bar{x})$ to be a different combination of $\sin \lambda\bar{x}$ and $\cos \lambda\bar{x}$ in each of the intervals $\bar{I}_1$ and $\bar{I}_2$. The coefficients of these combinations are determined by imposing the conditions

$$X(0) = X(\pi) = 0 \quad (5.3)$$

$$\frac{\partial X_-}{\partial \bar{x}} = \frac{\partial X_+}{\partial \bar{x}} = \bar{h}[X_+ - X_-]. \quad (5.4)$$

In order for there to be a nontrivial solution to the homogeneous equations, the determinant must vanish, yielding the "frequency equation" to be satisfied by λ. The corresponding coefficients and the functions X are then determined. Proceeding in this way one finds

$$\lambda = -\bar{h}[\tan \lambda\theta\pi + \tan \lambda\pi(1-\theta)] \quad (5.5)$$

and

$$X = \begin{cases} \sin \lambda\bar{x}, \; \bar{x} \in \bar{I}_1 \\ (\cos \lambda\theta\pi) \sin \lambda(\bar{x}-\pi) / \cos \lambda(1-\theta)\pi, \; \bar{x} \in \bar{I}_2, \end{cases} \quad (5.6)$$

where for each positive λ, the corresponding function X is determined to within a multiplicative constant.

The equation (5.5) defines an infinite sequence of positive roots λ which we denote in order of increasing size by $\lambda_1, \lambda_2, \ldots$. This statement, and the fact that the numbers λ_n approach infinity with increasing n, are readily deduced from consideration of the graphical interpretation of the roots of (5.5), $\bar{h}$ being a positive number. (An illustration for the case $\theta = 1/3$ is furnished by Fig. 3.) To each λ_n, termed an eigenvalue of the problem defined by (5.1), (5.3) and (5.4), there corresponds an eigenfunction $X_n(\bar{x})$.

At the same time, there corresponds to each λ_n a solution of (5.2) which is a combination of $\sin \lambda_n\tau$ and $\cos \lambda_n\tau$. Bearing in mind that the derivative with respect to τ is to vanish at $\tau = 0$, from (4.15) in the final solution, we shall assume the solution for $T(\tau)$ to be simply $\cos \lambda_n\tau$.

This then gives an infinity of particular solutions

$$X_n(\bar{x}) \cos \lambda_n\tau, \; n = 1, 2, \ldots, \quad (5.7)$$

of (4.13), (4.14), (4.16) and the second of conditions (4.15). Following the usual Fourier procedure we now satisfy the initial condition $r(\bar{x}, 0) = f(\bar{x})$ by choosing for $r(\bar{x}, \tau)$ the infinite series

$$r(\bar{x}, \tau) = \sum_{n=1}^{\infty} a_n X_n(\bar{x}) \cos \lambda_n\tau, \quad (5.8)$$

where a_n is the Fourier coefficient of $f(\bar{x})$ with respect to X_n in the expansion

$$f(\bar{x})=\sum_{n=1}^{\infty} a_n X_n(\bar{x})\,d\bar{x} \tag{5.9}$$

so that

$$a_n=\frac{\int_0^{\pi} f(\bar{x})X_n(\bar{x})\,d\bar{x}}{\int_0^{\pi} X_n^2\,d\bar{x}}\,,\quad n=1,2,\ldots. \tag{5.10}$$

In Appendix A it is shown that the eigenfunctions X_n may be characterized by a variational principle from which it follows that they satisfy the orthogonality relation

$$\int_0^{\pi} X_m X_n\,d\bar{x}=0,\quad m\neq n\,, \tag{5.11}$$

and possess the property of completeness. While it can be shown that certain classes of functions can be expanded in pointwise convergent series in the functions X_n, it is not necessary to prove the validity of (5.8) to obtain the final expression for the piston motion. This is shown in Appendix B. In essence, the "time" integral of $r(\bar{x},\tau)$ rather than $r(\bar{x},\tau)$ is expanded in the corresponding generalized Fourier series.

The solution for the reduced density variation function is then found from (4.12) to be

$$\bar{p}'(\bar{x},\tau)=f(\bar{x})-\sum_{n=1}^{\infty} a_n X_n(\bar{x})\cos\lambda_n\tau\,. \tag{5.12}$$

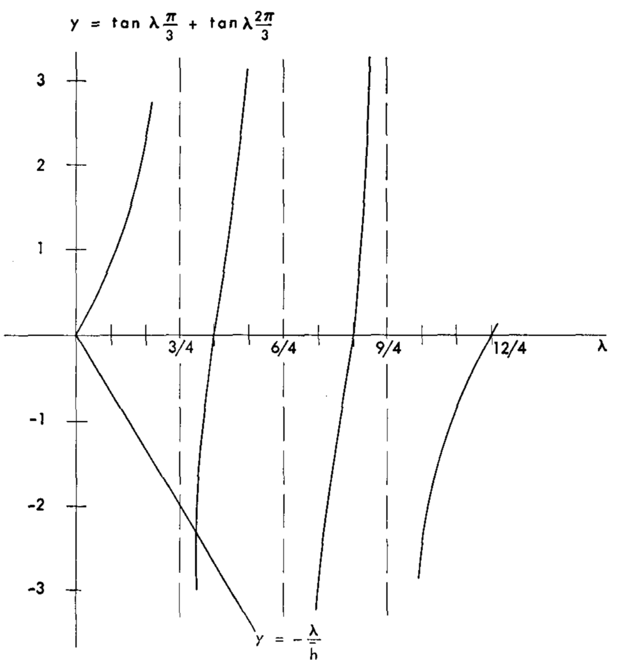

Figure 3 **Graph of frequency equation for case $\theta=1/3$.**

6. Motion of the piston

Let

$$\bar{\xi}=\frac{\pi\xi}{l_1+l_2} \tag{6.1}$$

denote the reduced piston displacement. Then in terms of nondimensional variables, the equation of motion (3.4) of the piston becomes

$$\frac{d^2\bar{\xi}}{d\tau^2}=\bar{h}[\bar{p}_-'-\bar{p}_+'] \tag{6.2}$$

where $\bar{h}$ is as before. The right-hand side of (6.2) is determined from (5.12) and (4.11) evaluating the right- and left-hand limits of $\bar{p}'$ and $f(\bar{x})$ at $\bar{x}=\theta\pi$. This gives

$$\frac{d^2\bar{\xi}}{d\tau^2}=\bar{h}\left[\frac{\bar{p}_1'-\bar{p}_2'}{\pi\bar{h}+1}+\sum_{n=1}^{\infty} a_n\cos\lambda_n\tau(X_n^+-X_n^-)\right],$$

where $X_n^{\pm}=\lim_{\bar{x}\to\theta\pi\pm0} X_n(\bar{x})$. From the jump condition (5.4) and the definition of X_n from (5.6),

$$h(X_n^+-X_n^-)=\lambda_n\cos\lambda_n\theta\pi$$

so that

$$\frac{d^2\bar{\xi}}{d\tau^2}=\frac{\bar{h}(\bar{p}_1'-\bar{p}_2')}{\pi\bar{h}+1}+\sum_{n=1}^{\infty} a_n\lambda_n\cos\lambda_n\theta\pi\cos\lambda_n\tau\,. \tag{6.3}$$

Integrating this function and making use of the initial conditions

$$\bar{\xi}(0)=\left.\frac{d\bar{\xi}}{d\tau}\right|_{\tau=0}=0,\quad \text{we obtain}$$

$$\bar{\xi}(\tau)=\frac{\bar{h}(\bar{p}_1'-\bar{p}_2')}{2(\bar{h}\pi+1)}\tau^2+\sum_{n=1}^{\infty}\frac{a_n}{\lambda_n}\cos\lambda_n\theta\pi-\sum_{n=1}^{\infty}\frac{a_n}{\lambda_n}\cos\lambda_n\tau\cos\lambda_n\theta\pi\,. \tag{6.4}$$

In Appendix B, this formally obtained expansion is shown to represent a true solution for the motion of the piston.

We proceed next to evaluate the coefficients a_n appearing above through Eq. (5.10), where $f(\bar{x})$ is given by (4.11) and X_n by (5.6) with λ replaced by λ_n. Carrying out the integrations gives

$$\int_0^{\pi} X_n^2\,d\bar{x}=\tfrac{1}{2}\left[\theta\pi+\pi(1-\theta)\frac{A_n^2}{B_n^2}+\frac{1}{\bar{h}}A_n^2\right] \tag{6.5}$$

and

$$\int_0^{\pi} fX_n\,d\bar{x}=\frac{1}{\lambda_n(1+\pi\bar{h})}\left[\bar{p}_1'(1+\pi\bar{h})+A_n\pi\bar{h}(\bar{p}_1'-\bar{p}_2')-\frac{A_n}{B_n}(\pi\bar{h}\bar{p}_1'+\bar{p}_2')\right], \tag{6.6}$$

where for convenience we set

$$A_n=\cos\lambda_n\theta\pi\,,\quad B_n=\cos\lambda_n\pi(1-\theta)\,. \tag{6.7}$$

The frequency equation (5.5) for λ_n can then be written

$$\lambda_n = \frac{-h \sin \lambda_n \pi}{A_n B_n}. \tag{6.8}$$

The Fourier coefficients of $f(\bar{x})$ with respect to X_n are therefore given by

$$a_n = \frac{2hB_n}{\lambda_n(1+\pi h)} \cdot \frac{[\bar{p}_1' B_n(1+\pi h) + A_n B_n h\pi(\bar{p}_1' - \bar{p}_2') - A_n(\pi h \bar{p}_1' + \bar{p}_2')]}{[B_n^2\, \theta\pi h + A_n^2\, B_n^2 + h\pi(1-\theta)A_n^2]}, \tag{6.9}$$

which completes the determination of the solution $\bar{\xi}(\tau)$.

7. Numerical results

Of particular interest is the case where the end pressures are equal so that $\bar{p}_1' = \bar{p}_2'$. In this case motion of the piston occurs if and only if the piston is initially positioned off center in the tube so that $l_1 \neq l_2$. By symmetry then one need only consider the cases for which $0 < \theta < 1/2$.

The Fourier coefficient simplifies to

$$a_n = \frac{2h\bar{p}_1'\, B_n(B_n - A_n)}{\lambda_n[B_n^2\, \theta\pi h + A_n^2\, B_n^2 + h\pi(1-\theta)A_n^2]} \tag{7.1}$$

and

$$\bar{\xi}(\tau) = 2h\bar{p}_1' \sum_{n=1}^{\infty} \frac{A_n B_n (B_n - A_n)(1-\cos \lambda_n \tau)}{\lambda_n^2[B_n^2\, \theta\pi h + A_n^2\, B_n^2 + h\pi(1-\theta)A_n^2]}. \tag{7.2}$$

As will be seen shortly, the terms in the series (7.2) are $0\left(\frac{1}{n^3}\right)$ as $n \to \infty$ due to the presence of the factor λ_n^{-2}, so that the convergence is rapid.

As an example, prior to discussing the general numerical results, we take the case where $l_2 = 2l_1$ so that $\theta = 1/3$. This can be carried through quite explicitly and illuminates the numerical aspects of the problem. The frequency equation becomes

$$\lambda_n = -h\left[\tan \lambda_n \frac{\pi}{3} + \tan \lambda_n \frac{2\pi}{3}\right]. \tag{7.3}$$

The eigenvalues λ_n are determined as the intersections of the line $y = -\frac{\lambda}{h}$ and the curve $y = \left[\tan \lambda\frac{\pi}{3} + \tan \lambda\frac{2\pi}{3}\right]$ in the $\lambda - y$ plane as shown in Fig. 3, where one period $0 \leq \lambda \leq 3$, is graphed. These intersections will occur in the neighborhood of the asymptotes of the functions $\tan \lambda_n \frac{\pi}{3}$ and $\tan \lambda_n \frac{2\pi}{3}$ and only there. As n increases, the intersections will approach the values of λ corresponding to these asymptotes. The asymptotes are at $\frac{3(2k-1)}{2}$ and $\frac{3(2k-1)}{4}$, $k = 1, 2, \ldots$, and if we denote the amount by which these numbers differ from the actual eigenvalues by ε_k' and ε_k'' respectively, we define two sequences of eigenvalues

$$\lambda_k' = \frac{3(2k-1)}{2} + \varepsilon_k'$$

and

$$\lambda_k'' = \frac{3(2k-1)}{4} + \varepsilon_k''. \tag{7.4}$$

The actual sequence of eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$, is obtained by ordering the λ_k' and λ_k'' into a single sequence of monotone increasing members.

Substituting for λ_k' from (7.4) in (7.3) and retaining $\varepsilon_k'^2$ terms in the expansions of the trigonometric terms leads to the error estimate

$$\varepsilon_k' = \frac{2h}{(2k-1)\pi},$$

so that the relative error in replacing λ_k' by $\frac{3(2k-1)}{2}$ is $\frac{4h}{3(2k-1)^2}$.

Similarly one can show that the relative error corresponding to ε_k'' is

$$0\left(\frac{1}{(2k-1)^2}\right).$$

In order to obtain explicit expressions to illustrate the present discussion we will take $\lambda_k' = \frac{3(2k-1)}{2}$ and $\lambda_k'' = \frac{3(2k-1)}{4}$. These approximations are clearly good for larger k, but may be used for all $k = 1, 2, 3, \ldots$, if we suppose h small (the physical significance of h is seen later).

The eigenvalues λ_n $n = 1, 2, \ldots$, are the successive numbers 3/4, 3/2, 9/4, 15/4, 9/2, 21/4, 27/4, 15/2, ..., in which for $n = 2, 5, 8, 11, \ldots$, the entries are successive members of the λ_k' sequence, while the entries for $n = 1, 3, 4, 6, 7, 9, 10, \ldots$ are successive members of the λ_k'' sequence. In order to compute the Fourier coefficients which enter into (7.2) it is necessary to evaluate for each λ_n the corresponding values of $A_n = \cos \frac{\lambda_n \pi}{3}$ and $B_n = \cos \frac{\lambda_n 2\pi}{3}$. From the frequency equation (7.3) it is possible to find explicit formulas for these quantities in terms of n and h to the same order of approximation as the values of λ_n.

Two sets of approximation formulas are obtained arising from the two subsequences of λ_n. We note that corresponding to the sequence λ_k', A_n will be near zero, whereas for the λ_k'' it is B_n, which is near zero. Specifically

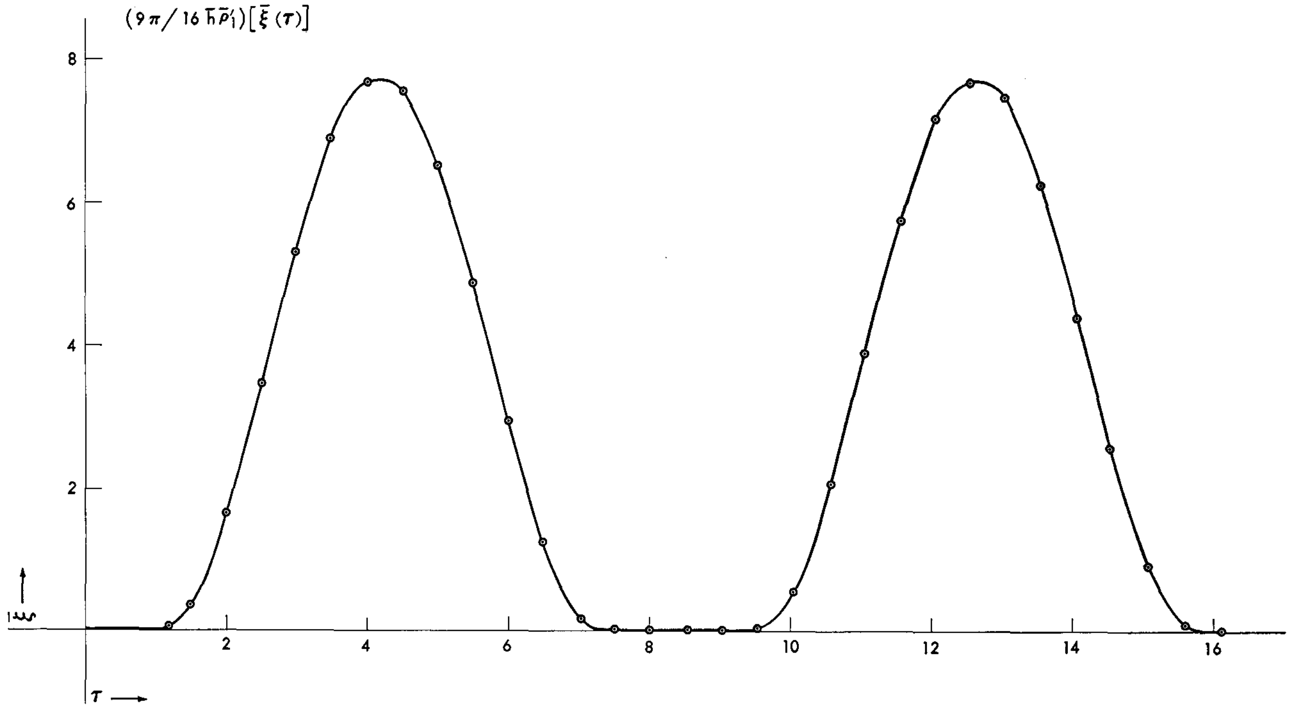

Figure 4 **Profile of piston displacement $\bar{\xi}$ vs τ for $\theta=1/3$ (asymptotic approximation for $\bar{h}$ small).**

$$\cos\lambda_k'\frac{\pi}{3}=\frac{2\bar{h}(-1)^k}{3(2k-1)}\,,\quad \cos\lambda_k'\frac{2\pi}{3}=-1$$
$$\cos\lambda_k''\frac{\pi}{3}=\pm\frac{\sqrt{2}}{2}\,,\quad \cos\lambda_k''\frac{2\pi}{3}=\frac{4(-1)^k h}{3(2k-1)}\,. \tag{7.5}$$

Substituting in (7.2) we obtain finally the displacement of the piston in the form

$$\bar{\xi}(\tau)=\frac{16\bar{h}\bar{p}_1'}{9\pi}\Big[4(1-\cos 3\tau/4)-(1-\cos 3\tau/2)-\frac{4}{3^3}(1-\cos 9\tau/4)+\frac{4}{5^3}(1-\cos 15\tau/4)+\frac{1}{3^3}(1-\cos 18\tau/4)-\dots\Big] \tag{7.6}$$

where the terms $n=2, 5, 8, \dots,$ in this series in brackets are the successive terms $k=1, 2, 3, \dots,$ of the sequence

$$\frac{(-1)^k}{(2k-1)^3}\left(1-\cos\frac{(2k-1)\tau}{4}\right),$$

and the remaining terms $n=1, 3, 4, 6, 7, \dots,$ are successive terms of the sequence

$$\frac{4(-1)^{k+1}}{(2k-1)^3}\left(1-\cos\frac{3(2k-1)\tau}{4}\right).$$

The series in brackets in (7.6) has been computed and is shown plotted in Fig. 4 for two cycles of the motion as the piston moves to the right and returns. It will be noted that no motion takes place until $\tau=\pi/3$, this being the reduced time required for the first wave to reach the piston at $\bar{x}=\pi/3$, the reduced velocity of sound being unity.

We return now to a numerical study of the piston motion in the case of equal end pressures $\bar{p}_1'=\bar{p}_2'$.

The motion of the piston is given by Equation (7.2). Therein the end pressure appears only in the multiplicative factor $\bar{p}_1'$. The parameters affecting the series are θ and h, which appear explicitly and also serve to determine the eigenvalues λ_n through the frequency equation, (6.8).

In terms of θ we have, from (4.5) and (4.10)

$$\bar{h}=\frac{\rho_0 l_1}{M\pi\theta}\,. \tag{7.7}$$

If we denote the cross-sectional area of the tube by A, then the mass of fluid to the left of the piston is $\rho_0 l_1 A$ and the mass of the piston itself is MA. Letting the ratio of these masses be m we have

$$\bar{h}=\frac{m}{\pi\theta}\,,\qquad m=\frac{\rho_0 l_1}{M}\,. \tag{7.8}$$

We shall deal with m and θ as representing the essential nondimensional physical parameters of the problem and relate to mass and geometry respectively. Fixing m and θ determines the quantity

$$F(\tau)=\bar{h}\sum_{n=1}^{\infty}\frac{A_nB_n(B_n-A_n)(1-\cos\lambda_n\tau)}{\lambda_n^2[B_n^2\,\theta\pi\bar{h}+A_n^2\,B_n^2+\bar{h}\pi(1-\theta)A_n^2]}\,, \tag{7.9}$$

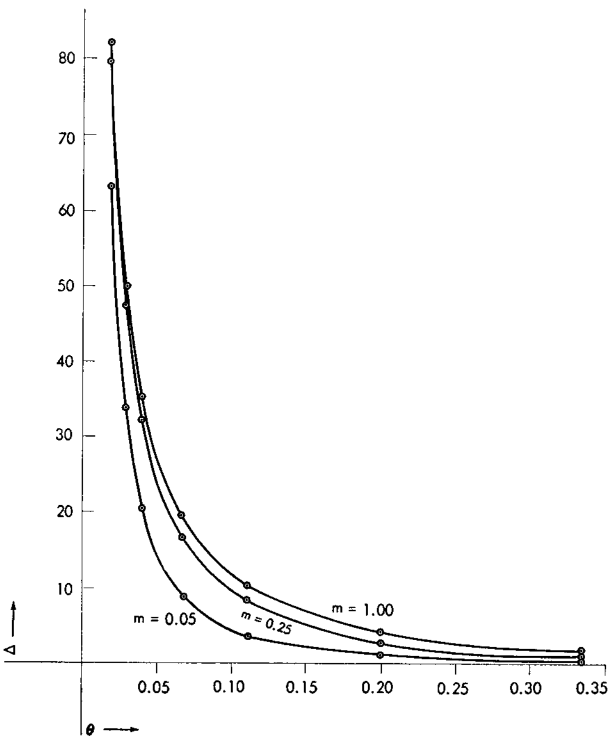

Figure 5 **Piston excursion as a function of θ.**

from which the actual displacement ξ of the piston is found from the relation

$$\xi = \frac{2\bar{p}_1' l_1}{\pi\theta} F(\tau) , \tag{7.10}$$

where we have used (7.2), (6.1) and (4.5). An appropriate nondimensional measure of the piston displacement in terms of the m and θ parameters is therefore the quantity

$$\xi' = \frac{F(\tau)}{\theta} = \frac{\pi}{2\bar{p}_1'} \frac{\xi}{l_1} . \tag{7.11}$$

In terms of τ and the parameter θ, the actual time t is given by

$$t = \frac{l_1}{a_0 \pi} \frac{\tau}{\theta} \tag{7.12}$$

from (4.1) and (4.5). Here, $\frac{l_1}{a_0}$ is the time required for a sound wave originating at the left end of the tube to reach the piston. As a nondimensional measure of time which includes the full dependence on the parameter θ we shall use the quantity

$$t' = \frac{\tau}{\theta} = \frac{a_0 \pi}{l_1} t . \tag{7.13}$$

The graph of Fig. 4 is typical of those of piston displacement versus time for any combination of m and θ values. The essential variation with m and θ lies in the "amplitude" of the wave and its "period." (The motion is found to be very near to periodic.) These quantities are the significant ones if it is desired to drive the piston hydraulically through a specified excursion (maximum initial displacement) in a specified time. We denote the excursion by Δ and the excursion time by T. For each pair of specified values of m and θ, ξ' was calculated as a function of τ and the maximum value of ξ' in the first wave was read off as Δ and the corresponding value of τ/θ as T.

The calculations were carried out on the IBM 704 with m values taken to be 0.05, 0.1, 0.25, 0.5, 0.75, 1, 2, and 5. For each m the following values of θ were used: 1/3, 1/5, 1/9, 1/15, 1/25, 1/35, 1/55. In all, this gave 56 runs. In each case Δ and T were determined. The results are depicted graphically in Figs. 5 to 8.

Figure 5 shows the variation of Δ with θ for several values of m. Figure 6 shows the variation of T with θ for two values of m. The variation with m is relatively small so that the graphs in these figures would have been excessively crowded if the other m values were included. To see the relative lack of dependence on m as compared to θ, in Fig. 7 the variation of Δ with m is plotted for the selected values of θ and in Fig. 8 the variation of T with m is plotted for these θ values.

By specifying $\bar{p}_1' = p_1'/p_0$, l_1 and a_0 in addition to m and θ, one can in any case calculate the actual maximum piston displacement ξ and required time t from (7.10) and (7.12). Conversely, the graphs may also be conveniently used to determine the required m and θ to achieve a specified Δ and T. For example, if T is given, using Fig. 6 one determines a range of θ values corresponding to varying m. Checking this range of θ values on Fig. 5 will show which, if any, m value will yield the desired Δ value.

A final word concerns the accuracy to which the calculations were carried out. The eigenvalues were calculated

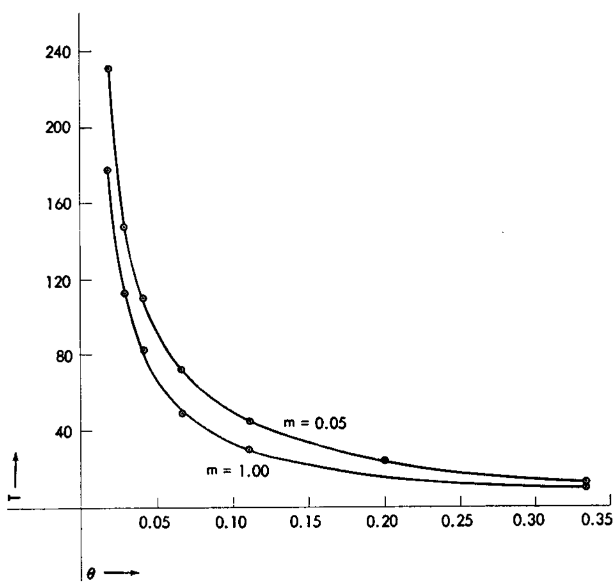

Figure 6 **Piston excursion time as function of θ.**

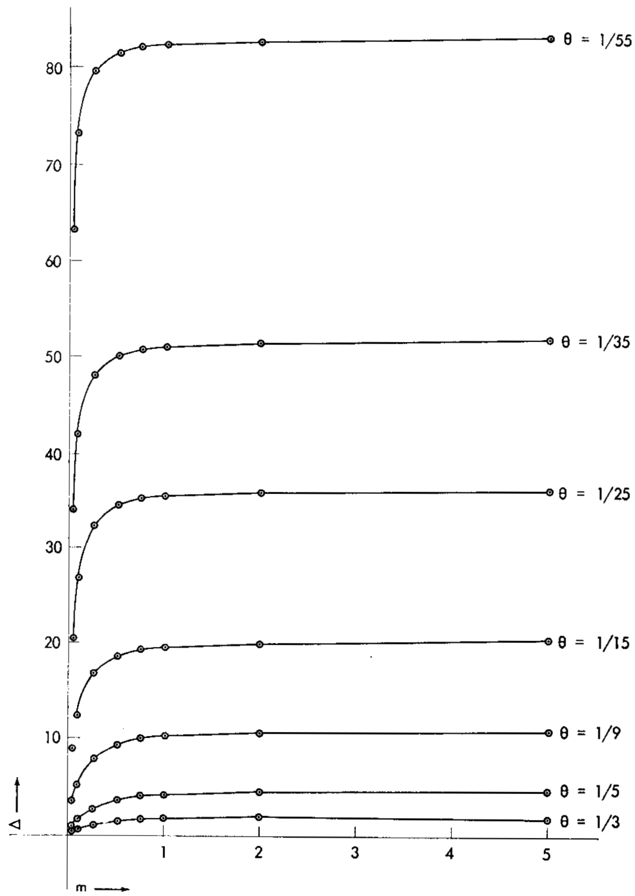

Figure 7 **Piston excursion as function of *m*.**

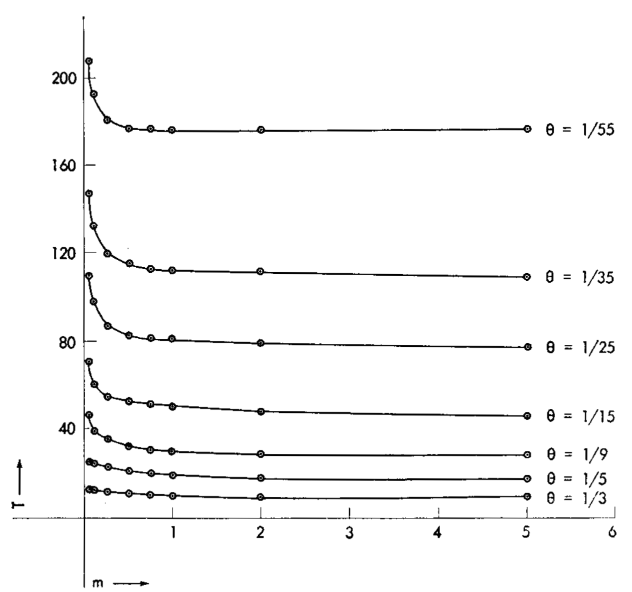

Figure 8 **Piston excursion time as function of *m*.**

from the frequency equation (5.5) by a Newton-Raphson iteration to an error less than 1×10^{-6}. The series of (7.9) was summed to approximately 50 terms, corresponding to a determination of all eigenvalues less than 50 in value and including the corresponding terms. The accuracy was checked by repeating, for extreme values of the parameters, the calculations using the series terms for all eigenvalues less than 100, essentially 100 terms. The observed change in Δ was nowhere large enough to affect the location of the points plotted in Figs. 5 and 7. For T, there was no change in the values for $r=0.05$ and $r=1.00$, and even at $r=5.00$ the change was of the order of 3%.

It will be noted that the points of Fig. 8 do not all lie exactly along smooth curves. This is due to the fact that $F(\tau)/\theta$ was calculated in intervals of $\Delta\tau=0.1$ and that the maximum $F(\tau)/\theta$ was chosen from among these values. Thus, the value of $T=\tau/\theta$ at this point can be in error (i.e., differ from the location of the true maximum) by as much as $0.05/\theta$ simply because of the coarseness of the $\Delta\tau$ interval and the lack of interpolation.

However, from (7.12) we see that this corresponds to an error in actual time of $\Delta t=\dfrac{l_1}{a_0\pi}\left(\dfrac{0.01}{\theta}\right)$, which may be several orders of magnitude smaller because of the size of a_0, the velocity of sound.

Appendix A: Orthogonality and completeness of the eigenfunctions

By showing that the eigenfunctions and eigenvalues determined by (5.1), (5.3) and (5.4) can be generated by minimizing an appropriate positive definite quadratic functional, we can by standard techniques deduce the needed properties of orthogonality and completeness for the eigenfunctions as well as the convergence of expansions in these eigenfunctions. We shall here only outline the procedure and specify certain results.

The variational problem to be considered is the following. Minimize

$$D[\phi]\equiv\int_0^\pi \phi'^2 d\bar{x}+a[\phi(\theta\pi+0)-\phi(\theta\pi-0)]^2 \qquad \text{(A-1)}$$

over the class of functions ϕ which are continuous over $0\leq\bar{x}\leq\pi$ except for a possible jump discontinuity at an interior point $\bar{x}=\theta\pi$, and which have piecewise continuous first and second derivatives over $0\leq\bar{x}\leq\pi$, subject to the following subsidiary conditions:

$$H[\phi]\equiv\int_0^\pi \phi^2 d\bar{x}=1\,, \qquad \text{(A-2)}$$

$$\phi(0)=\phi(\pi)=0\,, \qquad \text{(A-3)}$$

$$\phi'(\theta\pi+0)=\phi'(\theta\pi-0)\,, \qquad \text{(A-4)}$$

where $a>0$ is a constant.

The following results are established by proceeding as in [2], Chapter VI. The present problem represents a generalization of the functionals there considered by allowing for a discontinuity in the eigenfunctions themselves.

As necessary conditions for a minimum in the preceding problem one finds, denoting the minimizing function by ϕ_1 and the associated minimum by μ_1,

$$\phi_1''+\mu_1\phi_1=0\,, \quad 0\leq\bar{x}<\theta\pi\,, \quad \theta\pi<\bar{x}\leq\pi \tag{A-5}$$

and

$$\phi_1'(\theta\pi\pm0)=a[\phi_1(\theta\pi+0)-\phi_1(\theta\pi-0)]\,. \tag{A-6}$$

One then considers a sequence of minimum problems, in which in addition to the previous conditions the admissible functions are required to be orthogonal to the minimizing functions ϕ_i of the previous problems in the sense

$$\int_0^\pi \phi\phi_i d\bar{x}=0\,. \tag{A-7}$$

One thus obtains a sequence of orthonormal functions ϕ_i, $i=1,2,\ldots$, and associated numbers $\mu_1\leq\mu_2\leq\ldots$, satisfying (A-5) and (A-6) with 1 replaced by i. It follows that with $a=h$, the ϕ_i are (to within the normalizing constant factors) among the eigenfunctions defined by (5.1), (5.3), (5.4) with μ_i the associated eigenvalues.

That the sequences ϕ_i and μ_i constitute all the eigenfunctions and eigenvalues so defined, can then be proved by verifying that any eigenfunction not contained in the sequence ϕ_i must be orthogonal to the ϕ_i and hence, by a later expansion theorem, identically zero. Therefore we can write $\phi_i\equiv\alpha_i X_i$ and $\mu_i=\lambda_i$, $i=1,2,\ldots$, where α_i are the normalizing constants, and X_i and λ_i are defined by (5.5) and (5.6).

From the fact that $\lim_{i\to\infty}\lambda_i=\infty$ it follows, (see [2], §.3) that if $\psi(\bar{x})$ is any function satisfying the admissibility conditions of the original minimum problem, then

$$\lim_{n\to\infty}\int_0^\pi\left[\psi-\sum_{i=1}^n c_i\phi_i\right]^2 d\bar{x}=0\,, \tag{A-8}$$

where

$$c_i=\int_0^\pi \psi\phi_i d\bar{x} \tag{A-9}$$

is the Fourier coefficient of ψ with respect to ϕ_i, so that one has convergence in the mean. Equivalently

$$\int_0^\pi \psi^2 d\bar{x}=\sum_{i=1}^\infty c_i^2\,.$$

It can then be proved (see, e.g., [2], page 427) that the series $\sum_{i=1}^\infty c_i\phi_i$ converges uniformly and absolutely $0\leq\bar{x}<\theta\pi$, $\theta\pi<\bar{x}\leq\pi$, to the function ψ. A distinctive feature of this result in the present problem is that the series converges uniformly to a possibly discontinuous function, since ψ is permitted a discontinuity at $\bar{x}=\theta\pi$ (the value of the function and the series at $\bar{x}=\theta\pi$ is immaterial). This is due to the fact that the eigenfunctions ϕ_i themselves are discontinuous at this point.

This expansion theorem in itself does not provide the justification for the Fourier expansion (5.5) for the function $r(\bar{x},\tau)$, because $r(\bar{x},\tau)$ considered for fixed τ as a function of $\bar{x}$ violates the admissibility conditions in that it possesses in general discontinuities at points other than $\bar{x}=\theta\pi$. However, in order to justify the final result for the piston displacement we will see in Appendix B that this expansion theorem will suffice.

Appendix B: Validity of the solution

We shall here establish the validity of the series (6.4) for the piston motion. For simplicity we consider the case where $\bar{p}_2'=\bar{p}_1'$.

We consider first the function $r(\bar{x},\tau)$ which by definition satisfies the following conditions

$$\frac{\partial^2 r}{\partial\bar{x}^2}=\frac{\partial^2 r}{\partial\tau^2}\,,\ \bar{x}\in\bar{I}_1\cup\bar{I}_2,\quad \tau>0 \tag{B-1}$$

(excepting discontinuity lines)

$$r(0,\tau)=r(\pi,\tau)=0,\quad \tau>0 \tag{B-2}$$

$$r(\bar{x},0)\equiv1,\ r_\tau(\bar{x},0)=0,\ \bar{x}\in\bar{I}_1\cup\bar{I}_2 \tag{B-3}$$

$$\frac{\partial r_-}{\partial\bar{x}}=\frac{\partial r_+}{\partial\bar{x}}=h[r_+-r_-]\,. \tag{B-4}$$

We assume that a solution $r(\bar{x},\tau)$ to this problem exists which possesses two continuous derivatives and satisfies the equation (B-1) except along certain lines $\bar{x}\pm\tau$ = constant [characteristics of (B-1)] across which r and its normal derivative may be discontinuous, but the tangential derivative remains continuous. It will also be assumed that a discontinuity is propagated along a characteristic without change.

The existence of such a solution, following an argument communicated to the author by R. Courant, can be deduced from the representation of the solution in the form $r(\bar{x},\tau)=F(\bar{x}+\tau)+G(\bar{x}-\tau)$, using the initial and boundary conditions to continue the definitions of the functions F and G to all allowable values of the arguments $\bar{x}+\tau$ and $\bar{x}-\tau$. This argument, in effect, allows one to construct the solution $r(\bar{x},\tau)$ in a step-wise fashion and provides an alternative method of solution to the one presently under discussion. The details will not be presented here.

We now consider the function

$$R(\bar{x},\tau)=\int_0^\tau r(\bar{x},\tau)d\tau\,. \tag{B-5}$$

For any $\tau>0$, R is a continuous function of $\bar{x}$ for $\bar{x}\neq\theta\pi$ since in the range of integration the discontinuities of the integrand occur at most along a finite number of lines $\bar{x}\pm\tau$ = constant. At $\bar{x}=\theta\pi$ $R(\bar{x},\tau)$ will be discontinuous and it can be shown that

$$\frac{\partial R^-}{\partial\bar{x}}=\frac{\partial R^+}{\partial\bar{x}}=h[R_+-R_-] \tag{B-6}$$

for all $\tau\neq\tau_i$ where τ_i, $i=1,2,\ldots$, correspond to the successive times at which a discontinuity wave from the right or left end of the tube strikes the piston, i.e., reaches

$\bar{x}=\theta\pi$. (This will be proved as Lemma 1 at the conclusion of this Appendix.)

From (B-2) we have that

$$R(0,\tau)=R(\pi,\tau)=0,\quad \tau>0\,. \tag{B-7}$$

The function R for $\tau>0$, $\tau\neq\tau_i$ therefore satisfies all of the essential admissibility conditions of the variational principle and expansion theorem of Appendix A. Accordingly we can expand $R(\bar{x},\tau)$ for such τ in a uniformly and absolutely convergent series in the eigenfunctions X_n,

$$R(\bar{x},\tau)=\sum_{n=1}^{\infty}C_n(\tau)X_n(\bar{x})\,, \tag{B-8}$$

where

$$C_n(\tau)=\alpha_n^2\int_0^{\pi}R(\bar{x},\tau)X_n(\bar{x})d\bar{x}\,, \tag{B-9}$$

and $\alpha_n^{-2}=\int_0^{\pi}X_n^2d\bar{x}$. To complete the determination of $R(\bar{x},\tau)$ we need to obtain $C_n(\tau)$.

Formally from (5.8) and (B-5) we obtain for $R(\bar{x},\tau)$ the expansion

$$R(\bar{x},\tau)=\sum_{n=1}^{\infty}\frac{a_n}{\lambda_n}X_n(\bar{x})\sin\lambda_n\tau\,. \tag{B-10}$$

To validate this expansion we must establish then that

$$C_n(\tau)=\frac{a_n}{\lambda_n}\sin\lambda_n\tau,\quad \tau>0,\,\tau\neq\tau_i\,. \tag{B-11}$$

The coefficients a_n from (5.10) and (4.11) (since $\bar{p}_1'=\bar{p}_2'=1$ and therefore $f(\bar{x})\equiv1$ in this discussion) reduce to

$$a_n\equiv\alpha_n^2\int_0^{\pi}X_nd\bar{x}\,. \tag{B-12}$$

By Lemma 2, to be proved at the conclusion of this Appendix

$$C_n''(\tau)+\lambda_n^2C_n=0,\quad \tau>0,\,\tau\neq\tau_i\,. \tag{B-13}$$

It follows that in each interval $\tau_i<\tau<\tau_{i+1}$

$$C_n=p_n\cos\lambda_n\tau+q_n\sin\lambda_n\tau\,, \tag{B-14}$$

where p_n, q_n are constants which might vary for successive intervals in which the differential equation (B-13) holds. However, by (B-9) and the continuity of R we can verify that $C_n(\tau)$ and $C_n'(\tau)$ are both continuous functions for all $\tau>0$. It follows that p_n, q_n must be fixed constants independent of the interval of τ.

The constants p_n, q_n can now be determined from the initial conditions. We have by (B-7), (B-5) and (B-12)

$$|C_n'(\tau)-a_n|=\alpha_n^2\left|\int_0^{\pi}(r-1)X_nd\bar{x}\right|\,. \tag{B-15}$$

Since $r=1$ for $\tau=0$, and for sufficiently small τ, $r=1$ except for arbitrarily small intervals near $\bar{x}=0$ and π, and since X_n is bounded, it follows that $\lim_{\tau\to0}C_n'(\tau)=a_n$.

From (B-14) we then have

$$q_n=\frac{a_n}{\lambda_n}\,. \tag{B-16}$$

Similarly from the continuity of $R(\bar{x},\tau)$ at $\tau=0$ and the initial value $R(\bar{x},0)=0$ we find that

$$\lim_{\tau\to0}C_n(\tau)=0$$

so that in (B-14) we must have $p_n=0$. This establishes (B-11), and hence (B-10) for $R(\bar{x},\tau)$ is valid for $\tau\neq\tau_i$. From the continuity of $R(\bar{x},\tau)$ as a function of τ and the uniform convergence of the series in (B-10) of continuous functions of τ (this follows from the fact that the terms in (B-10) are $0\left(\frac{1}{n^2}\right)$) it follows that the expression (B-10) must be valid for the points $\tau=\tau_i$ as well.

Returning to the motion of the piston now, we have from (6.2) by integration, by the definitions (4.12) and (4.11),

$$\begin{aligned}\frac{d\xi}{d\tau}&=h\int_0^{\tau}[f_--f_+]d\tau-h\int_0^{\tau}[r_--r_+]d\tau\\&=h\,\frac{\bar{p}_1'-\bar{p}_2'}{\pi h+1}\,\tau-h[R_--R_+]\,.\end{aligned} \tag{B-17}$$

Substituting the series for R_- and R_+ in (B-17) obtained by using $X_n(\theta\pi-0)$ and $X_n(\theta\pi+0)$ from (5.6) in (B-10) and termwise integrating the resulting uniformly convergent series in τ, returns us to the expansion (6.4) for the piston motion which we set out to verify.

It remains to prove the two lemmas used above.

• *Lemma 1*

$$\frac{\partial R_\pm}{\partial\bar{x}}=h[R_+-R_-]\text{ for }\tau>0,\,\tau\neq\tau_i\,.$$

If τ_i denotes the successive τ values at which discontinuity waves reach the piston at $\bar{x}=\theta\pi$, and if $\bar{\tau}\neq\tau_i$, then for $\bar{x}$ sufficiently close to $\theta\pi$ on either the right or the left, there must be in the range $0<\tau<\bar{\tau}$ an even number of crossings of discontinuous waves corresponding to pairs of incoming and reflected waves on the given side of the piston. This is illustrated in Fig. 9, where one reflection is shown and the range of integration is split up into three subintervals by the points $\tau=a$ and $\tau=b$. The abscissa $\bar{x}$ corresponding to this line of integration approaches $\bar{x}=\theta\pi$ in the limit. Clearly $a=a(\bar{x})$ and $b=b(\bar{x})$ and it follows that

$$\begin{aligned}\frac{\partial R}{\partial\bar{x}}=&\int_0^{\bar{\tau}}\frac{\partial r}{\partial\bar{x}}d\tau+r(\bar{x},a-0)\frac{\partial a}{\partial\bar{x}}-r(\bar{x},a+0)\frac{\partial a}{\partial\bar{x}}\\&+r(\bar{x},b-0)\frac{\partial b}{\partial\bar{x}}-r(\bar{x},b+0)\frac{\partial b}{\partial\bar{x}}\,.\end{aligned} \tag{B-18}$$

By study of conservation of mass across the wave discontinuities meeting at the piston, it can be shown[3] that the reflected wave has the same discontinuity as the incident wave so that

$$r(\bar{x},a-0)-r(\bar{x},a+0)=r(\bar{x},b-0)-r(\bar{x},b+0). \quad \text{(B-19)}$$

At the same time $\dfrac{\partial a}{\partial \bar{x}}=-\dfrac{\partial b}{\partial \bar{x}}=1$, so that all terms on the right of (B-18) cancel except for the integral. Thus

$$\left.\frac{\partial R}{\partial \bar{x}}\right|_{\bar{x}=\theta\pi\pm 0}=\int_0^\tau \left.\frac{\partial r}{\partial \bar{x}}\right|_{\bar{x}=\theta\pi\pm 0} d\bar{x}, \quad \tau\neq\tau_i. \qquad \text{(B-20)}$$

From (B-4) and (B-5) the lemma then follows. It is also seen that if $\tau=\tau_i$, there will be an incident wave on either the right or left of $\bar{x}=\theta\pi$ whose contribution in terms of the jump in r is not cancelled by a reflected wave, as on the right side of (B-18), so that there will be an added term in (B-20).

• *Lemma 2*

$$C_n''(\tau)+\lambda_n^2 C_n(\tau)=0, \quad \tau>0, \tau\neq\tau_i.$$

We first verify that at points $(\bar{x},\tau)$ not lying on any of the discontinuity waves, $R(\bar{x},\tau)$ satisfies the wave equation

$$R_{\bar{x}\bar{x}}=R_{\tau\tau}. \qquad \text{(B-21)}$$

For simplicity let us assume that the τ segment from 0 to τ erected at $\bar{x}$ intersects only one discontinuity wave, say at $(\bar{x},a)$. The considerations will apply to any number of such intersections. As above, we then have

$$\frac{\partial R}{\partial \bar{x}}=\int_0^\tau \frac{\partial r}{\partial \bar{x}}d\tau+[r], \qquad \text{(B-22)}$$

where $[r]\equiv r(\bar{x}+0,a)-r(\bar{x}-0,a)$, and this formula is valid for discontinuities of either slope $+1$ or -1. Using the property that a discontinuity in r, a solution of the wave equation, travels unchanged along a characteristic, we have $\dfrac{\partial[r]}{\partial \bar{x}}=0$ so that

$$\frac{\partial^2 R}{\partial \bar{x}^2}=\int_0^\tau \frac{\partial^2 r}{\partial \bar{x}^2}d\tau+\left[\frac{\partial r}{\partial \bar{x}}\right], \qquad \text{(B-23)}$$

where $\left[\dfrac{\partial r}{\partial \bar{x}}\right]=\dfrac{\partial r}{\partial \bar{x}}(\bar{x}+0,a)-\dfrac{\partial r}{\partial \bar{x}}(\bar{x}-0,a)$. Replacing

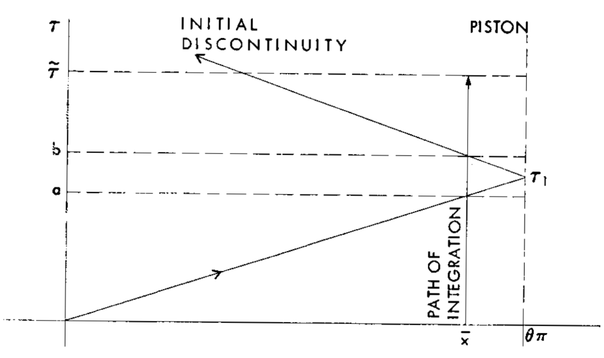

Figure 9 **Typical intersection of discontinuities and path of integral defining $R(\bar{x},\tau)$.**

$\dfrac{\partial^2 r}{\partial \bar{x}^2}=\dfrac{\partial^2 r}{\partial \tau^2}$ and breaking up the integral in (B-23) over the subintervals of continuity from 0 to a and from a to τ and using the initial condition $\left.\dfrac{\partial r}{\partial \tau}\right|_{\tau=0}=0$, we have

$$\frac{\partial^2 R}{\partial \bar{x}^2}=\frac{\partial r}{\partial \tau}+\gamma\left[\frac{\partial r}{\partial \tau}\right], \qquad \text{(B-24)}$$

where the γ is assigned the slope of the discontinuity. Now expressing $\dfrac{\partial r}{\partial \tau}$ and $\dfrac{\partial r}{\partial \bar{x}}$ in terms of the normal and tangential derivatives along the discontinuity characteristics and using the already assumed properties of $r(\bar{x},\tau)$ it can be shown that along a characteristic discontinuity

$$\gamma\left[\frac{\partial r}{\partial \tau}\right]=-\left[\frac{\partial r}{\partial \bar{x}}\right]. \qquad \text{(B-25)}$$

Combining the last three equations we find that

$$\frac{\partial^2 R}{\partial \bar{x}^2}=\frac{\partial r}{\partial \tau}, \qquad \text{(B-26)}$$

which is by the definition of R just $\dfrac{\partial^2 R}{\partial \tau^2}$. Thus (B-21) is satisfied for $\tau>0$, $x\in\bar{I}_1\cup\bar{I}_2$ provided the point $(\bar{x},\tau)$ does not lie on a discontinuity wave.

We now consider the equation

$$\int_0^\pi X_n(R_{\bar{x}\bar{x}}-R_{\tau\tau})d\bar{x}=0, \quad \tau>0. \qquad \text{(B-27)}$$

In the range of integration there will occur points of discontinuity of R and its derivatives due both to the intersection of discontinuity waves and the discontinuity at the piston $\bar{x}=\theta\pi$. Let $\tilde{x}$ denote one or the other of these points of discontinuity. Then integrating by parts for fixed $\tau>0$,

$$\int_0^\pi X_n R_{\bar{x}\bar{x}}d\bar{x}=\int_0^\pi RX_n''d\bar{x}+[RX_n']-[X_nR_{\bar{x}}], \quad \text{(B-28)}$$

where now $[\chi]\equiv\chi(\tilde{x}+0,\tau)-\chi(\tilde{x}-0,\tau)$. Since the position of $\tilde{x}$ may vary with τ we have

$$\frac{\partial}{\partial \tau}\int_0^\pi RX_nd\bar{x}=\frac{\partial}{\partial \tau}\left(\int_0^{\tilde{x}(\tau)}RX_nd\bar{x}+\int_{\tilde{x}(\tau)}^\pi RX_nd\bar{x}\right)$$

$$=\int_0^\pi X_n\frac{\partial R}{\partial \tau}d\bar{x}-[RX_n]\frac{d\tilde{x}}{d\tau}. \qquad \text{(B-29)}$$

In cases where $\tilde{x}$ refers to a discontinuity moving along a characteristic, both R and X_n are continuous so that the last term in (B-29) drops out. Differentiating again gives

$$\frac{\partial^2}{\partial \tau^2}\int_0^\pi RX_nd\bar{x}=\int_0^\pi X_n\frac{\partial^2 R}{\partial \tau^2}d\bar{x}-\left[\frac{\partial R}{\partial \tau}X_n\right]\frac{d\tilde{x}}{d\tau}. \qquad \text{(B-30)}$$

In this case $[RX_n']=0$ in (B-28) so that substituting from (B-28) and (B-30) in (B-27) gives

$$\int_0^\pi RX_n''\, d\bar{x} - [X_n R_{\bar{x}}] - \frac{\partial^2}{\partial\tau^2}\int_0^\pi RX_n d\bar{x} - \left[X_n \frac{\partial R}{\partial\tau}\right]\gamma = 0\,, \tag{B-31}$$

where $\gamma = \dfrac{d\bar{x}}{d\tau} = \pm 1$ as before. Since X_n is continuous at this discontinuity, the terms in brackets cancel by virtue of (B-25) applied to the solution R of the wave equation and we have

$$\frac{\partial^2}{\partial\tau^2}\int_0^\pi RX_n d\bar{x} - \int_0^\pi RX_n'' d\bar{x} = 0\,. \tag{B-32}$$

In the second case the discontinuity contribution is due to the presence of the piston. Here $\bar{x} \equiv \theta\pi$ and $\dfrac{d\bar{x}}{d\tau} = 0$. Thus,

$$\frac{\partial^2}{\partial\tau^2}\int_0^\pi X_n R d\bar{x} = \int_0^\pi X_n \frac{\partial^2 R}{\partial\tau^2}\, d\bar{x}\,. \tag{B-33}$$

Assuming now that $\tau \neq \tau_i$, we have that both $\dfrac{\partial R}{\partial\bar{x}}$ and X_n' are continuous at $\bar{x} = \theta\pi$ and in fact

$$X_n' = h[X_n],\ \frac{\partial R}{\partial\bar{x}} = h[R]\,. \tag{B-34}$$

In (B-28) we have, therefore,

$$[RX_n'] - \left[X_n \frac{\partial R}{\partial\bar{x}}\right] = X_n'[R] - \frac{\partial R}{\partial\bar{x}}\,[X_n]$$
$$= h[X_n][R] - h[R][X_n] = 0\,,$$

so that

$$\int_0^\pi X_n R_{\bar{x}\bar{x}}\, d\bar{x} = \int_0^\pi RX_n'' d\bar{x}\,. \tag{B-35}$$

Combining (B-33) and (B-35) again leads to (B-32). Thus, regardless of discontinuities we are led to (B-35), valid for $\tau > 0$, $\tau \neq \tau_i$. (A third case, where the discontinuity for given τ occurs at an end-point $\bar{x} = 0$ or $\bar{x} = \pi$ may be disposed of by the boundary conditions to give the same result.)

By (B-9)

$$\frac{\partial^2}{\partial\tau^2}\int_0^\pi RX_n d\bar{x} = \frac{1}{\alpha_n^2}\, C_n''(\tau)\,, \tag{B-36}$$

and of course

$$\int_0^\pi RX_n'' d\bar{x} = -\lambda_n^2 \int_0^\pi RX_n d\bar{x} = -\frac{\lambda_n^2}{\alpha_n^2}\, C_n\,. \tag{B-37}$$

Combining (B-22), (B-36) and (B-37) yields the statement of the lemma.

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