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Short-Length Menger Theorems

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Abstract

We give short and simple proofs of the following two theorems by Galil and Yu [3]. Let s and t be two vertices in an n-node graph G.
(1) There exist k edge-disjoint s-t paths of total length $O(n\sqrt{k})$.
(2) If we additionally assume that the minimum degree of G is at least k, then there exist k edge-disjoint s-t paths, each of length $O(n/k)$.

Let $G = (V, E)$ be an undirected n-node graph, with no parallel edges, and let s and t be two vertices of G such that there exist k edge-disjoint s-t paths. Our goal is to give short proofs of the following two theorems of Galil and Yu [3].

Theorem 1 *There exist k edge-disjoint s-t paths of total length $O(n\sqrt{k})$.*

Theorem 2 *If we additionally assume that the minimum degree of G is at least k, then there exist k edge-disjoint s-t paths, each of length $O(n/k)$.*

We view G as a directed graph by replacing each undirected edge by two oppositely oriented directed edges. Our proof of the first theorem is based on a maximum flow algorithm of Even and Tarjan [2]; for our purposes, we need only consider its global structure. The algorithm of [2] runs in phases numbered $1, 2, \ldots$. In phase d, a residual graph is maintained as a layered directed graph: the endpoints of each edge lie either in the same layer or in adjacent layers, and the distance from s to t is equal to d. The algorithm finds augmenting s-t paths of length d in this layered graph until there exist no more such paths of length at most d; the phase then ends.

Proof of Theorem 1. We analyze the behavior of the Even-Tarjan algorithm for producing a flow of value k in G. We prove an upper bound on the total length of all augmenting paths found; this also upper bounds the total length of the flow paths. We set $\ell = 2nk^{-1/2}$. We say that an augmenting path is of type 0 if its length is at most ℓ, and of type i ($i \geq 1$) if its length is between $2^{i-1} \cdot \ell$ and $2^i \cdot \ell$. The total length of all type 0 paths is at most $k\ell = 2n\sqrt{k}$. To bound the total length of all type i paths, for $i \geq 1$, note that at the start of phase $2^{i-1} \cdot \ell$ of the Even-Tarjan algorithm, there is some pair of adjacent layers in the residual graph whose union contains at most $n/(2^{i-2} \cdot \ell)$ vertices. Between this pair of layers there can be at most $n^2/(2^{2i-2} \cdot \ell^2)$ edges, and hence at most this many augmenting paths can be produced from phase $2^{i-1} \cdot \ell$ onward. Thus the total length of all type i paths is at most $(n^2 2^{-2i+2} \ell^{-2}) \cdot (2^i \ell) = 4n^2 \ell^{-1} 2^{-i} = 2n\sqrt{k}2^{-i}$, and so the total length of *all* augmenting paths is at most $2n\sqrt{k} + 2n\sqrt{k}\sum_{i\geq 1} 2^{-i} = 4n\sqrt{k}$. ■

For the proof of the second theorem, we consider the problem of finding a set of k edge-disjoint s-t paths in G whose total length is minimum. Let $P_1, \ldots, P_k$

be such a set of paths; say that an edge e is a *flow edge* if it is contained in some P_i, and a *non-flow edge* otherwise. Observe that if (u, v) is a flow edge, then by the optimality of $P_1, \ldots, P_k$, (v, u) is not a flow edge. Finding k edge-disjoint paths of minimum total length is a minimum-cost flow problem; if we take its linear programming dual, we obtain dual variables y_v, one for each $v \in V$, which are integers with the following properties. (See e.g. [1].)

(1) If (u, v) is a flow edge, then $y_v - y_u \geq 1$.

(2) If neither (u, v) nor (v, u) is a flow edge, then y_u and y_v differ by at most 1.

(3) We may assume without loss of generality that $y_s = 0$, $y_t \geq 0$, and for every j in the interval $[0, y_t]$, there exists a node v with $y_v = j$. We define $X_i = \{v : y_v = i\}$.

Proof of Theorem 2. We claim that for each $i \in [0, y_t - 3]$, $\left|\cup_{j=i}^{i+4} X_j\right| \geq \frac{1}{5}k$; the theorem will follow directly from this. By (3), there exist vertices $u \in X_{i+1}$, $v \in X_{i+2}$, and $w \in X_{i+3}$. Now, suppose that at most $\frac{1}{5}k$ flow paths pass through at least two of them. At least $3k - 3$ edges are incident to $\{u, v, w\}$; at most $4 \cdot \frac{1}{5}k + 2 \cdot \frac{4}{5}k = \frac{12}{5}k$ of these are flow edges. Hence there are at least $\frac{3}{5}k - 3$ non-flow edges incident to $\{u, v, w\}$; at least $\frac{1}{5}k - 1$ of these are incident to a single one of these vertices. By (2), the endpoints of these non-flow edges lie in $\cup_{j=i}^{i+4} X_j$, and hence $\left|\cup_{j=i}^{i+4} X_j\right| \geq \frac{1}{5}k$.

Otherwise, at least $\frac{1}{5}k$ flow paths pass through at least two of u, v, w. Now, by (1), at most one flow path can pass through both u and v or both v and w; and at most one can pass directly from u to w. Thus (again by (1)), at least $\frac{1}{5}k - 3$ of these paths must pass from u to w via distinct vertices in X_{i+2}. Hence $|X_{i+2}| \geq \frac{1}{5}k - 3$, and so again $\left|\cup_{j=i}^{i+4} X_j\right| \geq \frac{1}{5}k$. ■

References

[1] R. Ahuja, T. Magnanti, J. Orlin, *Network Flows*, Prentice-Hall, 1993.

[2] S. Even, R. Tarjan, “Network flow and testing graph connectivity,” *SIAM J. Computing*, 4(1975), pp. 507–518.

[3] Z. Galil, X. Yu, “Short-length versions of Menger’s theorem,” *Proc. 27th ACM STOC*, 1995, pp. 499–598.