On the Foundations of Classical Planning: Some Algebraic and Topological Structures

Éric Jacopin*
Laforia-IBP, Boîte 169,
Université Pierre et Marie Curie,
4, place Jussieu,
F-75252 Paris Cedex 05
jacopin@laforia.ibp.fr
ftp://ftp.ibp.fr/ibp/softs/laforia/pweak/Pweak.html
Fax is +33 1 44 27 70 00

Abstract

This paper is an attempt to introduce mathematical structures for classical planning. It is first shown that the set-based structures underlying classical planning are unable to build more than a monoid. Consequently, an appropriate structure is presented (that of the incidence algebra of a partial plan) and used to build the partial plan space as an algebraic field. A vector space, an affine space and utlimately a metric space is build over the partial plan space. Plan modification operators operate as translations on the partial plan vectors. The structures are then applied to rewrite the mapping function used for the comparison of two partial plan spaces as a bijective and continuous function.

1 Introduction

This paper is an attempt to introduce a theory in the domain of artificial intelligence classical planning. It describes an ongoing and clearly unfinished research which begins to show some results.

The difference between a theory and a model comes from the nature of the reification process. A model describes the elements of the field its of study: operators and states of the problem, branching factor of the search space, etc; whereas a theory is interested in the relations (e.g. addition, multiplication, composition, etc) between the elements of the field and their properties (e.g. commutativity, associativity, etc). The idea behind theories is that of foundations: building structures to go further; and although there are as many types of elements as we can imagine there are a few properties of their relations. The idea behind thus is to minimize the materials to build a structure whereas models face the problem of finding the good (and next) parameter to better describe objects: "Give me three parameters I'll draw an elephant. Give me four and I'll make it walk".

What could a theory for classical planning look like?

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If partial plan space (PPS) classical planning was to solve a differential equation, it would not do it analytically. It would not solve the equation and then come up with the equation of the function satisfying the differential equation. On the contrary, PPS classical planning would try to numerically build up the graph of the function, starting from a point it knows it belongs to the graph (which could correspond to the initial partial plan) of the function and then trying to reach "another point" (the partial plan which is a solution of the problem). Moreover, it would try to solve the equation in a discrete space. Strangely, however, this "another point" is not located: rather, we know the properties of such a point (i.e. for classical planning it's a solution; and for differential equation solving, it ends up the construction of the graph of the function), but PPS classical planning does not know where it is. Indeed, PPS classical planning builds up a path in the partial plan space, possibly a dead-end path, until it reaches the solution node: the graph of the function solving the differential equation is built up from the start point, looking for the best possible nearest neighbor, until one of these neighbor is the solution node. And, for instance, there is no known method to jump over the nearest neighbor, to a node at a further distance. And there is no known method for partitionning the search in disjoint spaces as well. Eventually PPS classical planning would return the "another point" of the graph of the function it computed.

Any model tries to characterize the numerical solving. Any theory would instead try providing a framework where the analytical solving is possible.

Where to start?

If there was a rational answer, then it would probably define a general scientific method for discovering new theories; and it is clear this is yet to come. Moreover, there exists no theoretical background (logic is a descriptive tool, not a structural one) and even no previous attempt (I am aware of). Therefore, it has been chosen to introduce classical algebraic structures.

However, a quick analysis of PPS classical planning allows to understand that the usual PPS materials (plans and plan modifications) do not allow to build a(n algebraic) theory. It is the purpose of the next section to present this analysis and show the constraint preventing from an algebraic theory.

Section 3 contains algebraic structures for classical planning which are then extended to topological ones. This section makes PPS classical planning an affine space where the plans are vectors and where PPS plan modification operators acts as the group of translations on the vectors. Topological arguments rely on the distance of the translations.

Finally, an application is given in section 4; it contains steps towards a formal treatment of the comparison of the search space of the UA and TO planners [3]. It is shown in terms of the isomorphism of their respective partial plan space (when seen as a topological space): a continuous bijection over two topological spaces has been introduced as a condition for the isomorphism between these space. Note that from a mathematical point of view, this is a trivial point of homeomorphism between topological spaces; but from the classical planning point of view and considering the lack of any theoretical support, this point is, to me, a breakthrough.

I shall certainly not claim that the structures presented in this paper *are* the foundations that are needed. Rather, I would argue they offer some of the epistemological properties that foundations should possess. For instance, they should not only circumscribe the field in term of providing virtual (i.e. mathematical) beasts (i.e. structures) that can correspond to classical planning objects (e.g. plan, partial plan space, refinement operator, etc), but also elaborate directions of research for classical planning issues (such as parallelism, density of solutions, specialized partial plan space movements, etc).

The reader of this paper should be familiar with PPS classical planning and in particular [3]. Although the next sections are full of algebraic tools, they are all classics or so. Anyway, algebraic structures are detailed in the appendix, or else in the paper when needed.

2 A starting point

Solving a (classical planning) problem is refining an initial plan until it is a solution. A plan is a tuple of sets, each characterizing a property of a lattice of partially instantiated operators. In its simplest description, a plan is merely a partially ordered set (but still a lattice); i.e. a 2-tuple made of a set (of operators) and a partial order relation over this set, with a minimum and a maximum. Note the partial order is a subset of the cartesian product of the set of operators. Then the elements of a plan are sets. Therefore the refinement process toward the construction of a solution can only use set operations: $\cup$ (set-union), $\setminus$ (set-difference), etc. Moreover, this refinement process is an order-preserving process; that is, any precedence constraint of the partial order shall not be modified by the refinement process. If it happens the plan cannot be further refined towards what would be desired as being a solution, it is then abandoned (there happens the backtracking): there is no "undo" mechanism in the refinement process but instead the abandonment of dead end refinements[1].

Consequently, a plan modification operator (*pmo*) is an order-preserving function over PPS, which uses set-union to modify partial plans:

$$\begin{array}{rcccl} pmo & : & PPS & \rightarrow & PPS \\ & & p & \mapsto & q = pmo(p) = p \cup (O, <) \end{array}$$

Of course, in the general case, it is possible to consider $(O, <)$ as another partial plan (because the only demand on a partial plan so far is that it is a lattice of operators). No more than a quick glance, let us illustrate as a *pmo* the so-called establishment and demotion (O_i and O_f respectively are the minimum and maximum of the lattice p):

New establishment (of a precondition) of an operator A by another operator E (when E does not belong to p):

$$q = p \cup (\{O_i, E, A, O_f\}, \{(O_i, E), (E, A), (A, O_f)\})$$

Demotion of an operator C before an operator E (which is possible if and only if partial plan p is such that $\neg(E < C)$ is true):

$$q = p \cup (\{O_i, C, E, O_f\}, \{(O_i, C), (C, E), (E, O_f)\})$$

Of course, considering, for instance, $(\{O_i, C, E, O_f\}, \{(O_i, C), (C, E), (E, O_f)\})$ as a partial plan can eventually be puzzling. Mainly because current plan construction techniques may reject such a plan as a *partial* one (i.e. there is no path from the initial partial plan to this partial plan, using any current planning procedure). But again, in the general case one can see PPS as a (infinite) set of all possible lattices over a given set (of operators) such that O_i and O_f respectively, always

[1] It is intriguing to realize this exactly is the description of the **Poset** category (see [2] for categorical arguments on the structures here described).

are the minimum and maximum of the lattices; and where a plan construction criterion follows an order-preserving path, starting at $\{(O_i, O_f), (O_i, O_f)\}$.

However, the *crucial* point here is the *idempotency* of $\cup$: $\{a, b\} \cup \{a, b\} = \{a, b\}$ and not $\{a, b, a, b\}$. This prevents from seeing $(PPS \setminus (\emptyset, \emptyset), \cup)$ more than a monoid. Indeed $\cup$ is associative; and $ipp = \{(O_i, O_f), (O_i, O_f)\}$ can be seen as the unique identity element in $PPS^* = PPS \setminus (\emptyset, \emptyset)$. But there is no possible inverse[2] p^{-1} for a partial plan such that: $p \cup p^{-1} = p^{-1} \cup p = ipp$.

Therefore, as long as set-union is the basic tool to progress in PPS towards a solution, theoretical algebraic structures such as a group, a field, a vector space, etc, cannot be constructed: the use of $\cup$ has to be relaxed, and (consequently) so has the use of a set as a basic element of a plan-tuple.

What is proposed here, however, is not to (fully) reject the basic representation of a partially ordered set but to index this representation with the elements of what is already an algebraic field (see the appendix). This is the core of the next section.

3 Theoretical structures

First, it must be noted that this section is a mathematical presentation of classical planning, and not a new (at least different) view of planning. Indeed, the mathematical structures here presented do give structures for classical planning and "something else". As well as the set of points of the graph of a function, solution of a differential equation, both belongs to the graph of the function (of course) and "something else", e.g. $\mathbb{R}^2$. Sticking to this analogy, this section shows how to build such *a* "$\mathbb{R}^2$" for classical planning.

Since a theory is presented, structures are first given and only after what they refer to.

The partial plan space is here constructed as a (possibly discrete) affine space where the points are the partial plans; plan modification operators thus is the group of translations of the affine space whose origin is chosen to be the initial partial plan. This is described in the first subsection.

The second subsection adds metrics to the affine space (thus building a topological space), related to the plan modifications operators, thus allowing to compare the visit of the partial plans spaces (now affine spaces) through the modifications defined by some plan construction criterion[3] (pcc). If the pcc is the same for two partial plan spaces but differs from the fact that one is that of the totally ordered partial plans of the other (the case of the UA and TO planning procedures, for instance), then any *metrics* introduced within the pcc shall be crucial in the comparison. This shall be examined in the second subsection, as an application of the structures here presented.

3.1 Algebraic structures

We first study the notion of a partial plan and in second operations over partial plans are studied. Both are presented as an algebraic field.

Finally, an affine structure emerges from the relations between the structure of the partial plan space and the structure of the partial plan "operations".

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[2]This also prevents from using multisets, because the natural numbers, and not integers, are the base for multisets. Of course, one could think of p^{-1} as the complementary set of p; but then ipp is no more the unique unit element.

[3]For a pcc to be a plan "truth" criterion, there should be a mathematical tool constructed from the geometrical (affine) space to some "truth" space. This is clearly another story, but such a mathematical beast exists and is a topos (see [2]).

3.1.1 The incidence algebra of a partial plan

Let O henceforth be a set, O^2 be its associated partial order[4] and $\mathbb{F}$ be a field.

The previous section concluded that the use of $\cup$ prevented from having inverses of partial plans. One may then wish to look for mathematical representation of a partial plan that would allow inverses. Luckily, the inverse of a partial order has been studied through Möbius functions in combinatorial theory [1, 2]. We are then to use this representation, which is called the incidence algebra of a partially ordered set.

Then, the incidence algebra of the partially ordered set O with entries in $\mathbb{F}$ is noted I(O,$\mathbb{F}$) and is that of the functions over a partially ordered set returning 0 only when two elements are not comparable:

$$\mathrm{I}(\mathsf{O},\mathbb{F}) = \{f | f : \mathsf{O}^2 \rightarrow \mathbb{F}, \text{ such that if } (x,y) \notin \mathsf{O}^2 \text{ then } f(x,y) = 0\}$$

First note that in the case where O is finite, then I(O,$\mathbb{F}$) is a subalgebra of the algebra of upper triangular matrix of order of the cardinal of O and entries in $\mathbb{F}$ [1, page 140]. An upper triangular matrix is a common representation for a partial order.

Applications to Classical Planning Henceforth, O is a finite set (of planning operators) with cardinality $|\mathsf{O}|$; and $\mathbb{F}$ is a field that may be finite (such as $\mathbb{Z}/p\mathbb{Z}$ the set of consecutive integers from 0 to $p-1$ with p prime) or not (such as $\mathbb{R}$ the set of reals).

Definition 3.1 (Partial plan space) *Let I(O,$\mathbb{F}$) the incidence algebra of the locally finite partially ordered set O. I(O,$\mathbb{F}$) is the* partial plan space *of $\mathsf{O}$and is noted $\mathcal{PP}(\mathbb{F})$.*

Note $\mathcal{PP}(\mathbb{F})$ is considered for a fixed set of operators.

At first, we here do no more than looking at a partial plan where the partial order is syntactically represented with a multiset which semantically represents the number of times a precedence constraints between two operators has been modified, either by the so-called establishment or the declobbering process. However, the classical notion of multiset maps set elements to the positive integers whereas it is here to any field.

We shall henceforth omit mention of the field $\mathbb{F}$ and briefly write $\mathcal{PP}$, with the tacit convention that $\mathbb{F}$ is to remain fixed throughout, but as least as big as $\mathbb{Z}/2\mathbb{Z}$ (i.e. $|\mathbb{F}| \geq 2$). The use of an infinite field, such as $\mathbb{R}$ would mean that a real number could describe a precedence constraint between two operators. This obviously could be used for uncertainty purposes.

Definition 3.2 (Depth of the partial plan space) *Let $\mathcal{PP}$ be a partial plan space over a finite set O. The* depth *of $\mathcal{PP}$ is the cardinal of O and is noted $|\mathsf{O}|$.*

The depth of a partial plan space only bounds the number of operators in O, and does not bound in itself the number of elements of $\mathcal{PP}$ that are to be visited. Indeed, although the number of preconditions to be be achieved is obviously related to the number of operators in a partial plan, the number of elements of $\mathcal{PP}$ is finite only if its related field is finite.

[4] Theoretically, the partial order of O has to be locally finite, i.e. all the intervals of the partial order are finite. This always the case for classical planning partial plans where there is only a finite number of operators (possibly 0) between any two operators.

Since we now have an algebra, we get operations that, of course, do not mean anything for classical planning, but are necessary for the existence of the algebra. For instance, scalar multiplication or partial plan multiplication (what does the multiplication of 2 partial plans mean?) do not have any corresponding plan modification operation in classical planning. One should see this from two viewpoints.

First, it is a normal consequence of the introduction of a theory: the points of the graph of a function in $\mathbb{R}$ do represent something for the function: its graph. The points in $\mathbb{R}$ that do not belong to this graph *do not* represent anything for the function. But the graph *do* exist as such in $\mathbb{R}$.

Second, there is much to be done. Scalar or partial plan multiplications are new tools we could take advantage of; for instance for defining faster movements towards the solution.

Let's now examine what are the actual plan modification operators.

3.1.2 The field of plan modifications operators

Let $\varepsilon(\mathcal{PP})$ be the set of endomorphisms of $\mathcal{PP}$, that is, the set of mappings from $\mathcal{PP}$ to itself. From the informal discussion of section 2, it is clear that plan modification operators are elements of $\varepsilon(\mathcal{PP})$. Let $+$ and $\circ$ be the two usual binary operations on $\varepsilon(\mathcal{PP})$:

$$\begin{array}{ccccl} + & : & \varepsilon(\mathcal{PP})^2 & \rightarrow & \varepsilon(\mathcal{PP}) \\ & & (f,g) & \mapsto & f+g \text{ such that } \forall p \in \mathcal{PP},\ (f+g)(p) = f(p)+g(p) \end{array}$$

$$\begin{array}{ccccl} \circ & : & \varepsilon(\mathcal{PP})^2 & \rightarrow & \varepsilon(\mathcal{PP}) \\ & & (f,g) & \mapsto & g \circ f \text{ such that } \forall p \in \mathcal{PP},\ (g \circ f)(p) = g(f(p)) \end{array}$$

It is a basic result that $(\varepsilon(\mathcal{PP}),+,\circ)$ is a ring.

Let $1_n(\mathbb{F})$ be the subset of $\mathrm{I}(\mathsf{O},\mathbb{F})$ such that $f(x,x)=0$ for all x of O:

$$1_n(\mathbb{F}) = \{f | f : \mathsf{O}^2 \rightarrow \mathbb{F},\ \forall x \in \mathsf{O}\ f(x,x)=0\}$$

Note that $1_n(\mathbb{F})$ corresponds to irreflexive partial orders.

$1_n(\mathbb{F})$ is not a subalgebra of $\mathrm{I}(\mathsf{O},\mathbb{F})$(because the elements are no more inversible for the multiplication); however, $1_n(\mathbb{F})$ is an $\mathbb{F}$-module. Moreover, we note 1_{ij} an element of $1_n(\mathbb{F})$ whose functions are all 0, except one which is 1: $1_{ij} = f(i,j) = 1$ and $\forall x \neq i$ and $y \neq j, f(x,y) = 0$.

1_{ij} corresponds to a partial order with only one precedence constraint; for instance, the one the establishment or declobbering process is about to add.

It is immediate that the set of all possible 1_{ij}s is a basis for $1_n(\mathbb{F})$; it is noted $\{1_{ij}\}$[5].

Applications to Classical Planning We consider the ring of endomorphisms over $\mathcal{PP}$ (the partial plan space over O). In particular, we consider the following endomorphism (which, in terms of vector space, is a translation — the set of translations is an abelian subgroup of the permutations of the vector space which is isomorphic to the additive group of the vector space):

$$\begin{array}{ccccl} t_b & : & \mathcal{PP} & \rightarrow & \mathcal{PP} \\ & & p & \mapsto & t_b(p) = p+b \text{ such that } b \in \mathcal{PP} \end{array}$$

[5]The basis of $\mathcal{PP}$ can be written $\{1_{ij}\} \cup \{1_{11}, \ldots, 1_{|\mathsf{O}||\mathsf{O}|}\}$ and $\mathcal{PP}$ is isomorphic to $\mathbb{F}^{\frac{|\mathsf{O}|(|\mathsf{O}|+1)}{2}}$, and from a theoretical point of view, one should now only talk of $\mathbb{F}^{\frac{|\mathsf{O}|(|\mathsf{O}|+1)}{2}}$. However, to avoid further abstractions, we'll stick to $\mathcal{PP}$ for this paper.

Definition 3.3 (Basic plan modification operator) *Let $\mathcal{PP}$ be a partial plan space over a finite set* O*. A* basic plan modification operator *is a translation t_b by vector b such that $b \in \{1_{ij}\}$.*

We note $\mathcal{BRO}$ the set of all basic plan modification operators, which corresponds to all the refinements adding 1 to a precedence constraint. We can now consider bigger addition:

Definition 3.4 (General plan modification operator) *Let $\mathcal{PP}$ be a partial plan space over a finite set* O*. A* general plan modification operator *is a translation t_g by vector g such that $g \in 1_{|\mathsf{O}|}$.*

Property 3.1 (Set of general plan modification operators) *$\mathcal{GRO}$ the* set of classical planning general plan modification operators *is a subring of $\varepsilon(\mathcal{PP})$.*

Any translation is also an automorphism (i.e. a bijective endomorphism), so $(\mathcal{GRO}, \circ)$ is an abelian group, isomorphic to the abelian (additive) group $(\mathcal{PP}, +)$. Therefore,

Property 3.2 *$(\mathcal{GRO}, +, \circ)$ is a commutative field.*

3.1.3 The affine geometry of partial plans

Since $\mathcal{PP}$ is an algebra, $(\mathcal{PP}, +)$ is an $\mathbb{F}$-vector space; and as just noted, it is also isomorphic to $(\mathcal{GRO}, +)$ which is the abelian group of the translations over $\mathcal{PP}$. We then get the following:

Theorem 3.1 (The affine space of partial plan) *Let $\mathcal{PP}$ be a partial plan space over the finite set* O*; and let $\mathcal{GRO}$ be the set of general refinement operators of $\mathcal{PP}$. $\mathcal{PP}$ is an* affine space *over* O*.*

An affine space has a trick: it does not possess any particular origin (because translations operate on any point, it's alway possible to translate the origin to any point) whose definition is left out to suit any needs:

Definition 3.5 (Origin of $\mathcal{PP}$) *Let $\mathcal{PP}$ be an affine space over* O*. $1_{1|\mathsf{O}|}$ is the affine* origin *of $\mathcal{PP}$ and is noted $\mathcal{O}_{\mathcal{PP}}$.*

$1_{1|\mathsf{O}|}$ just corresponds to what was noted *ipp* in section 2, namely the classical initial partial plan, containing a precedence constraint only between the initial and final operator.

Consequently, we consider the following affine basis (a basis define the basic vectors with which any movement shall be written):

Definition 3.6 (Basis of $\mathcal{PP}$) *Let $\mathcal{PP}$ be an affine space over* O*. $(\mathcal{O}_{\mathcal{PP}}, \{1_{ij}\} \cup \{1_{11}, \ldots, 1_{|\mathsf{O}||\mathsf{O}|}\})$ is the affine* basis *of $\mathcal{PP}$ and is noted $\mathcal{B}_{\mathcal{PP}}$.*

This basis is that of moving (i.e. translating) along one precedence constraint at a time.

Applications to classical planning As well as the classical definition where the initial partial plan encodes the planning problem, the point of the affine space representing *ipp* is the planning problem of the affine space:

Definition 3.7 (Planning problem) *Let $\mathcal{PP}$ be an affine space over* $\mathtt{O}$ *together with its affine basis $\mathcal{B}_{\mathcal{PP}}$. The* planning problem *of $\mathcal{PP}$ is its affine origin $\mathcal{O}_{\mathcal{PP}}$.*

Definition 3.8 (Plan modification operator) *Let $\mathcal{PP}$ be a partial plan space over a finite set* $\mathtt{O}$*; and let $\mathcal{RO}$ be a subset of $\mathcal{GRO}$. A* plan modification operator *is a function which yields a set of translated points accordingly with all the element of $\mathcal{RO}$:*

$$\begin{array}{rcccl} \langle\!\langle\rangle\!\rangle_{\mathcal{RO}} & : & \mathcal{PP} & \to & P(\mathcal{PP}) \\ & & p & \mapsto & \langle\!\langle\rangle\!\rangle_{\mathcal{RO}}(p) = \{q | q = t_{ro}(p),\ \forall ro \in \mathcal{RO}\} \end{array}$$

In terms of classical planning, we have:

Definition 3.9 (Candidate set of a partial plan) *Let $\mathcal{PP}$ be a partial plan space over a finite set* $\mathtt{O}$*; and let $\mathcal{RO}$ be a subset of $\mathcal{GRO}$. Let p be a partial plan and $\langle\!\langle\rangle\!\rangle_{\mathcal{RO}}$ be a plan modification operator. $\langle\!\langle\rangle\!\rangle_{\mathcal{RO}}(p)$ is the* candidate set *of the partial plan p and is noted $\langle\!\langle p\rangle\!\rangle_{\mathcal{RO}}$.*

Definition 3.10 (Completeness of a plan modification operator) *A surjective plan modification operator is said to be* complete.

Definition 3.11 (Systematicity of a plan modification operator) *An injective plan modification operator is said to be* systematic.

We won't discuss surjectivity and injectivity of a plan modification operator any further. However, one still have to introduce a final concept, that of plan construction criterion. Plan modification operators *refine* a partial plan. But, up to now, we have no tool to direct the refinements that can be applied to a partial plan.

Definition 3.12 (Plan construction criterion) *Let $\mathcal{PP}$ be a partial plan space over a finite set* $\mathtt{O}$ *and $\mathcal{GRO}$ be its related set of general plan modification operators. A plan construction criterion is a function which yields a subset of $\mathcal{RO}$ from a partial plan ($\mathcal{P}(\mathcal{RO})$ is the set of partitions of $\mathcal{RO}$):*

$$\begin{array}{rcccl} \kappa & : & \mathcal{PP} & \to & \mathcal{P}(\mathcal{RO}) \\ & & p & \mapsto & \{t | t \in \mathcal{RO}\} \end{array}$$

When $\kappa(p) = \{t_{\mathbf{0}}\}$, then the plan construction process shall be "stationary". Moreover, in the (very) general case where translations over a vector space are considered, there always potentially exists a translation which can be applied to an *inconsistent* partial plan (which could be such that, for instance, $\langle\!\langle p\rangle\!\rangle_{\mathcal{RO}} = \emptyset$) on which classical planning would backtrack. But backtracking now merely is applying the inverses of the translations that were previously applied.

Definition 3.13 (Set of solutions of a planning problem) *Let $\mathcal{PP}$ be an affine space over* $\mathtt{O}$*. The* set of solutions of the planning problem *of $\mathcal{PP}$ is the kernel of the plan construction criterion:*

$$ker(\kappa) = \{p | \kappa(p) = \{t_{\mathbf{0}}\}\}$$

3.2 Topological structures

We here use a partial plan as a matrix whose rows and columns are indexed by the elements of O with entries in $\mathbb{F}$ (recall the algebra of the upper triangular matrices is a subalgebra of I(O,$\mathbb{F}$) [1]).

A partial plan is a matrix whose rows and columns are indexed by the elements of O with entries in $\mathbb{F}$. Thus, let us consider the following bilinear form over $\mathcal{PP}$:

$$\begin{array}{rcccl} \langle , \rangle_{\mathbb{F}} & : & \mathcal{PP} \times \mathcal{PP} & \rightarrow & \mathbb{F} \\ & & (p, q) & \mapsto & \langle p, q \rangle_{\mathbb{F}} = \sum_{i=1}^{|\mathsf{O}|} \sum_{j=1}^{|\mathsf{O}|} (p \cdot q)_{ij} \end{array}$$

Where $(p \cdot q)$ is the matrix multiplication of p by q[6].

When $\mathbb{F}$=$\mathbb{R}$, then $\langle p, p \rangle_{\mathbb{R}} = p^2$ and then $\|p\| = \sqrt{\langle p, p \rangle_{\mathbb{R}}}$ is the *norm* of the partial plan p.

Indeed, $\delta(p, q) = \|p - q\| = \sqrt{\langle (p - q), (p - q) \rangle_{\mathbb{R}}}$ has the properties of a distance. First $\delta(p, q) = 0$ only if $p = q$. Then, $\delta(p, q) = \delta(q, p)$ because of the commutativity of both matrix multiplication and real addition. Finally, $\delta(p, q) + \delta(q, r) \geq \delta(p, r)$ because equality only holds when $q = 0$.

Then,

Property 3.3 (Metric (partial plan) space) *Let $\mathcal{PP}(\mathbb{R})$ be an affine space over O together with its affine basis $\mathcal{B}_{\mathcal{PP}}$. $(\mathcal{PP}(\mathbb{R}), \delta)$ is a* metric space.

Applications to classical planning It is now possible to measure the distance of any partial plan from the origin of the space. This leads to the following definition:

Definition 3.14 (Depth of a partial plan) *Let $\mathcal{PP}$ be an affine space over O. The* depth *of a partial plan p is $\delta(\mathcal{O}_{\mathcal{PP}}, p)$.*

Classical plan generation is such that $\forall q \in \langle\langle p \rangle\rangle_{\mathcal{RO}}$ there exists $\bar{\kappa} \in \mathbb{R}^+$, such that $\delta_{\mathbb{F}}(p, q) < \bar{\kappa}$. Thus limiting the depth of the extension of a partial plan according to a pcc that is based on a local reasoning eventually establishing *only one*[7] precondition at each cycle of a planning procedure. Consequently, $\langle\langle p \rangle\rangle_{\mathcal{RO}}$ is an open ball of diameter $2\bar{\kappa}$.

A topology on a space serves to define continuous functions there. One can then use the topological structure induced by the metric, which can be based on the number of modifications made by a planning procedure to compare partial plan spaces. The next section gives an application in the case of the UA and TO planning procedures [3].

4 An application to classical planning

This section presents an application of the mathematical structures of the partial plan space that have been built in the previous section to classical planning.

In the previous section, the partial plan space has been structured as an affine space, whose points (or vectors) are partial plans and whose (group of) translations are plan modification operators.

[6]This matrix multiplication does nothing else than simulating a vector product in the configuration space $\mathbb{F}^{\frac{|\mathsf{O}|(|\mathsf{O}|+1)}{2}}$.

[7]Both UA and TO saturate their unachieved precondition set at each of their cycle, thus possibly establishing more than one precondition in one cycle. However, the establishment process is the same for every precondition in that one precondition at a time is handled.

Moreover, translations, by the number of modifications they apply to a partial plan, can be used to introduce a distance and thus the partial plan space eventually is a metric (partial plan) space. This section shows how to use the topological structure of a partial plan space so as to compare two partial plan spaces; and ultimately to compare the planning procedures that visits one space to the other procedure.

The structures are here used to compare the search space of the UA and TO planning procedures [3]. The reader should be familiar with this work to read what follows.

The comparison of UA and TO relies on a function, noted $\mathcal{L}$, which maps the partial plan space of a planning procedure (UA) which generates partially ordered plans to a partial plan space of another planning procedure (TO) which generates totally ordered plans. In [3], the planning procedures are first given, and only then the building of $\mathcal{L}$, which must respect two properties, in order to help a comparison of the search space of the planning procedures, both seen as a tree[8] and respectively named $tree_{UA}$ and $tree_{TO}$:

1. **Totality Property**: For every plan U in $tree_{UA}$, there exists a non-empty set $\{T_1, \ldots, T_m\}$ of plans in $tree_{TO}$ such that $\mathcal{L}(U) = \{T_1, \ldots, T_m\}$.
2. **Disjointness Property**: $\mathcal{L}$ maps distinct plans in $tree_{UA}$ to disjoint sets of plans in $tree_{TO}$; that is, if $U_1, U_2 \in tree_{UA}$ and $U_1 \neq U_2$, then $\mathcal{L}(U_1) \cap \mathcal{L}_(U_2) = \emptyset$.

Disjointness property is nothing else but the injectivity of $\mathcal{L}$. However, totality property is not surjectivity.

Now being now aware of the topological properties of a partial plan space, the question is: "Given two partial plan spaces such that one contains the totally ordered partial plans of the other, what are the constraints on the planning procedures visiting each spaces so that the set of nodes they visit are comparable?"

Using the mathematical structures of the previous sections, the answer is immediate: "the constraints derive from the homeomorphism of the search spaces". That is, making the spaces topologically equivalent shall induce some constraints on the planning procedures visiting them.

Two metric spaces are homeomorphic if and only if there exists a function f between those two spaces such that (i) it is a bijection and (ii) both f and f^{-1} are continuous (continuity is defined below).

We are to show that $\mathcal{L}$ can be this homeomorphism. For this, we are to change the totality property into surjectivity and then define the constraints on the planning procedures so that $\mathcal{L}$ is continuous. Note that despite the totality property is not surjectivity, the homeomorphic property is exactly what Minton, Bresina and Drummond designed to compare the search spaces.

Making $\mathcal{L}$ surjective means $\mathcal{L}$ is such that for every set $\{T_1, \ldots, T_m\}$ of plans in $tree_{TO}$, there exists a plan U in $tree_{UA}$ such that $\mathcal{L}(U) = \{T_1, \ldots, T_m\}$.

As one can see, surjectivity is not that different from totality property. Why then surjectivity is not used in [3]? Because the aim of $\mathcal{L}$ is to count the nodes of $tree_{TO}$ with regards to the number of nodes of $tree_{UA}$. Therefore, [3] starts from showing that for every U there always corresponds some set of Ts.

[8] The group of translations of an affine space, i.e. the set of plan modification operators acting over the space of partial plan, is a free group and indeed the graph of a free group is a tree [4].

But surjectivity can make the desired counting anyway. If there exists a bijection between two sets E and F then $|E| = |F|$ (see [1]). In our case, E is $tree_{UA}$ and F is $\mathcal{L}$'s domain: that of the *corresponding linearizations* of a plan U in $tree_{UA}$.

Let's precisely rewrite $\mathcal{L}$ ($\mathcal{P}(tree_{TO})$ is the set of partitions of $tree_{TO}$):

$$\begin{array}{rcll} \mathcal{L} & : & tree_{UA} & \rightarrow & \mathcal{P}(tree_{TO}) \\ & & U & \mapsto & \{T | T \in tree_{TO},\ \delta_{tree_{TO}}(\mathcal{O}_{tree_{TO}}, T) = \delta_{tree_{UA}}(\mathcal{O}_{tree_{UA}}, U)\} \end{array}$$

Of course, the distance is the same, according to the same set of asserted preconditions and the Ts are linearizations of U.

Let's remark that $\mathcal{L}(U)$ is never empty and let $MBD(tree_{TO})(\subset \mathcal{P}(tree_{TO}))$ stands for $\mathcal{L}$'s domain exactly. Since we want $\mathcal{L}$ to be a bijection, then $|tree_{UA}| = |MBD(tree_{TO})|$, which *immediately* entails $|tree_{UA}| \leq |tree_{TO}|$[9].

To get $\delta_{tree_{TO}}(\mathcal{O}_{tree_{TO}}, T) = \delta_{tree_{UA}}(\mathcal{O}_{tree_{UA}}, U)$, Minton, Bresina and Drummond makes δ the geometrical distance defined over a tree [4]: that of the vertices. This is *very* important because from a planning procedure point of view, the vertices are obtained in *one* cycle of the procedure (what [3] calls a 1-step extension). Consequently, the equality of the distances imposes to both procedures to extend a node in the same way.

Making $\mathcal{L}$ continuous on $tree_{UA}$ is making $\mathcal{L}$ such that for every U, V and $U \neq V$ in $tree_{UA}$, for every possible $\varepsilon > 0$, there exists $\alpha > 0$ such that[10]:

$$\delta_{tree_{UA}}(U, V) < \alpha \Rightarrow \delta_{\mathcal{P}(tree_{TO})}(\mathcal{L}(U), \mathcal{L}(V)) < \varepsilon$$

For ε to be strictly greater than 0, then $\mathcal{L}(U)$ and $\mathcal{L}(V)$ are different sets of partial plans; then (because $\mathcal{L}$ is bijective) U and V are different partial plans. And therefore α exists and is greater than 0.

For L to be an homeomorphism, $\mathcal{L}^{-1}$ also it to be continuous:

$$\delta_{\mathcal{P}(tree_{TO})}(T, S) < \alpha \Rightarrow \delta_{tree_{UA}}(\mathcal{L}^{-1}(T), \mathcal{L}^{-1}(S)) < \varepsilon$$

The reasoning is analoguous to the previous one. Again, for ε to be strictly greater than 0, then $\mathcal{L}^{-1}(T)$ and $\mathcal{L}(S)$ are different partial plans; then, (because $\mathcal{L}$'s bijectivity entails $\mathcal{L}^{-1}$'s) T and S are different sets of partial plans. And therefore α exists and is greater than 0.

Consequences on the planning procedures are that one must ensure they do a 1-step extension at each cycle ($\alpha > 0$ and $\varepsilon > 0$). For this to be true, one must be sure that the 1-call to the planning procedure does not leave a partial plan unchanged: one must add an operator at each cycle or else fail.

Since the reason for the addition of an operator is the establishment of a precondition, the cyclic addition of an operator is only possible when all the preconditions that could be established from operators already belonging to the plan have effectively been established.

[9] In fact, proving $\mathcal{L}$ injective is enough for counting: if there exists an injection between E and F then $|E| \leq |F|$ (see [1]), which entails the uselessness of the totality property for counting. However, we here need an homeomorphism; that's why $\mathcal{L}$ needs to be a bijection.

[10] Note that the distance $\delta_{\mathcal{P}(tree_{TO})}$ here is between sets of partial plans. Such a distance is *not* the distance δ making $\mathcal{PP}$ a metric space. Indeed δ is defined between two partial plans and not between two sets of partial plans. This latter is the classical hausdorff distance and uses the distance δ to be computed: $\delta_{\mathcal{P}(tree_{TO})}(\mathcal{L}(U), \mathcal{L}(V)) = \min(\{d | \mathcal{L}(U) \subset \mathcal{L}(V)_d$ and $\mathcal{L}(V) \subset \mathcal{L}(U)_d\})$ where $\mathcal{L}(\_)_d = \{p \in tree_{TO} | \exists q \in \_\ \delta(p, q) < d\}$.

From a TO point of view, plans are totally ordered. Then preconditions are else (necessarily) true or not (necessarily) true.

Due to the partial order of plans of UA some preconditions are false in some linearizations of these plans. This might be corrected later on in introducing a new operator or else new precedence constraints. But this violates the injectivity of $\mathcal{L}$. Consequently, one must also ensure a partial plan of $tree_{UA}$ established preconditions are also established in its $\mathcal{L}$-image. Hence the definition of the *interaction* between to operators of a plan in [3].

Finally, the planning procedures must (i) introduce a new operator each time and (ii) UA must reorder interacting steps. The contents of the procedures is then obvious.

The 1-step-extension based distance relies on the geometric property of a tree: that there is no connected path in tree and then vertices can do the job. This is as counting a distance in terms of blocks in a town and expecting every block has the same size.

This here is sufficient in terms of operator addition and the necessity to maintain what [3] calls *unambiguous* plans.

But this does not tell us anything about the size of the block: what is the number of modifications that both procedures should do?

One could choose some distance and thus circumscribe the number of modifications of a partial plans at each cycle. One could go a step further in linking the distance to a truth degree and thus exactly design the `interacts?` metric (which *only then* becomes a metric); but this is another story.

5 Conclusions

The partial plan space of classical planning has been shown to be an affine space whose translations are the plan modification operators. As an application, the mapping function helping the comparison of the UA and TO planning procedures has been show as an homeomorphism between the respective metric space of the planning procedures. This gives a general hint on the comparison of planning procedures: the distance of a partial plan to its origin is related to the number of plan modifications that have been applied. Counting these modifications and finding why and then they are applied is a path to the comparison.

But the theoretical beasts that has been presented asks much more questions than they answer. For instance, they mostly are geometrical and should also be “logical” so as to take into account the meaning of a “truth” criterion. Although they do encompass new plan construction criteria (with much bigger $\bar{\kappa}$, for instance) and new kind of movements in the partial plan spaces (i.e. not only translations but also rotations, for instance) they do not give any hint yet on how to build them. Again probably because there is no tool yet to indicate the (partial) truthness of a partial plan. But these are add-ons to what has been presented; not different backbones.

A An Appendix on Algebraic Structures

Definition A.1 (Binary Operation) $\star$ *is a* binary operation *of a set* S *if and only if* $\forall r, s \in S, (r \star s) \in S$.

Definition A.2 (Commutativity) $\star$ *is a commutative binary operation of a set* S *if and only if* $\forall r, s \in S, r \star s = s \star r$.

Definition A.3 (Associativity) *$\star$ is an associative binary operation of a set S if and only if $\forall r, s, t \in G,\ (r \star s) \star t = r \star (s \star t)$*

Definition A.4 (Monoid) *A* monoid *is a non empty set M together with a binary operation, $\star$ and the following properties:*

- *$\star$ is associative,*
- *there exists a unique $e \in G$ such that $\forall r \in G, e \star r = r \star e = e$; e is the identity element of G.*

Definition A.5 (Group) *A* group *is a non empty set G together with a binary operation, $\star$ and the following properties:*

- *$(G, \star)$ is a monoid,*
- *$\forall r \in G$ there exists a unique $s \in G$ such that $r \star s = s \star r = e$. s is noted r^{-1} and is the inverse of r.*

Moreover, a group G is said to be abelian (or commutative) if its binary operation is commutative. A group is a monoid for which all elements possess a unique inverse.

Definition A.6 (Ring) *A* ring *is a non empty set R together with two binary operation, usually noted $+$ and $\cdot$, and the following properties:*

- *$(R, +)$ is an abelian group,*
- *$\cdot$ is associative,*
- *$\forall r, s, t \in R,\ (r + s) \cdot t = r \cdot t + s \cdot t$ and $t \cdot (r + s) = t \cdot r + t \cdot s$.*

If $\cdot$ is commutative then the ring R is said to be commutative. If $(R, \cdot)$ is a monoid then R is a ring with identity.

Definition A.7 (Field) *A* field *is a non empty set $\mathbb{F}$ together with two binary operation, usually noted $+$ and $\cdot$, and the following properties:*

- *$(\mathbb{F}, +)$ is an abelian group,*
- *$(\mathbb{F} \setminus \{\mathbf{0}\}, \cdot)$ is a group,*
- *$\forall r, s, t \in R,\ (r + s) \cdot t = r \cdot t + s \cdot t$ and $t \cdot (r + s) = t \cdot r + t \cdot s$.*

If $\cdot$ is commutative then the field $\mathbb{F}$ is said to be commutative; any finite field (i.e. whose cardinal is finite) is commutative. Note that a field is nothing else but a ring where all elements but $\mathbf{0}$ have inverses (with $\cdot$).

Definition A.8 (Vector Space) *Let G be an abelian group and $\mathbb{F}$ be a commutative field. G is a* vector space *over $\mathbb{F}$ if there exists a mapping from $G \times \mathbb{F} \to G$, noted by juxtaposition, for which the following properties hold $\forall \alpha\beta \in \mathbb{F}$ and $\forall u, v \in G$:*

- *$\alpha(u + v) = \alpha u + \alpha v$,*

- $(\alpha + \beta)u = \alpha u + \beta v$,
- $\alpha(\beta u) = (\alpha\beta)u$,
- $\exists \mathbf{1} \in \mathbb{F}$ *such that* $\mathbf{1}u = u$.

The elements of G are referred to as vectors.

Definition A.9 (Algebra) *Let $\mathcal{A}$ be an non empty set and $\mathbb{F}$ be a commutative field. $\mathcal{A}$ is said to be an algebra if the following properties hold:*

- *$\mathcal{A}$ is a $\mathbb{F}$ vector space,*
- *$(\mathcal{A}, +, \cdot)$ is a ring,*
- *$\forall u, v \in \mathcal{A}$ and $\forall \alpha \in \mathbb{F}$, $\alpha(u \cdot v) = (\alpha u) \cdot v = u \cdot (\alpha v)$.*

Thus, an algebra is a vector space in which we can take the product of vectors, or a ring in which we can multiply each element by a scalar (subject, of course, to additional requirements, as given in the definition).

For further results and the properties presented in this paper, see:

Steve Roman *Advanced Linear Algebra*, Graduate Texts in Mathematics **135**, Springer Verlag (1992), 363 pages.

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