On the Extension of Pseudo-Boolean Functions for the Aggregation of Interacting Criteria

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Abstract

The paper presents an analysis on the use of integrals defined for non-additive measures (or capacities) as the Choquet and the Šipoš integral, and the multilinear model, all seen as extensions of pseudo-Boolean functions, and used as a means to model interaction between criteria in a multicriteria decision making problem. The emphasis is put on the use, besides classical comparative information, of information about difference of attractiveness between acts, and on the existence, for each point of view, of a "neutral level", allowing to introduce the absolute notion of attractive or repulsive act. It is shown

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that in this case, the Šipoš integral is a suitable solution, although not unique. Properties of the Šipoš integral as a new way of aggregating criteria are shown, with emphasis on the interaction among criteria.

Keywords: multicriteria decision making, Choquet integral, capacity, interactive criteria, negative scores

1 Introduction

Let us consider a decision making problem, of which the structuring phase has led to the identification of a family $\mathcal{C} = \{C_1, \ldots, C_n\}$ of n fundamental *points of view* (*criteria*), which permits to meet the concerns of the decision maker (DM) in charge of the above mentioned (decision making) problem. We suppose hereafter that, during the structuring phase, one has associated to each point of view C_i, $i = 1, \ldots, n$, a *descriptor* (*attribute*), that is, a set X_i of reference levels intended to serve as a basis to describe plausible impacts of potential actions with respect to C_i.

We make also the assumption that, for all $i = 1, \ldots, n$, there exists in X_i two particular elements which we call "Neutral$_i$" and "Good$_i$", and denoted $\mathbf{0}_i$ and $\mathbf{1}_i$ respectively, which have an absolute signification: $\mathbf{0}_i$ is an element which is thought by the DM to be neither good nor bad, neither attractive nor repulsive, relatively to his concerns with respect to C_i, and $\mathbf{1}_i$ is an element which the DM considers as good and completely satisfying if he could obtain it on C_i, even if more attractive elements could exist on this point of view. The practical identification of these absolute elements has been performed in many real applications, see for example [6, 8, 9].

In multicriteria decision aid, after the structuring phase comes the evaluation phase, in which for each point of view C_i, *intra-criterion* information is gathered (i.e. attractiveness for the DM of the elements of X_i with respect to point of view C_i), and also, according to an aggregation model chosen in agreement with the DM, *inter-criteria* information. This information, which aims at determining the parameters of the chosen aggregation model, generally consists in some information on the attractiveness for the DM of some particular elements of $X = X_1 \times \cdots \times X_n$. These elements are selected so as to enable the resolution of some equation system, whose variables are precisely the unknown parameters of the aggregation model.

In this paper, of which aim is primarily theoretical, we adopt with respect to the classical approach described above, a rather converse attitude. Specifically, we do not suppose to have beforehand a given aggregation model, but rather to have some information concerning the attractiveness for the DM of a particular collection of elements of X. Then we study how to extend this information on the preference of the DM to all elements of X. This kind of problem can be called an *identification of an aggregation model* which is compatible with available information.

The paper is organized as follows. In section 2, we introduce the basic assumptions we make concerning the knowledge on the attractiveness for the DM of particular elements of X. Section 3 shows that this kind of information is compatible with the existence of some interaction phenomena between points of view, and introduces some definitions related to the concept of interaction. The problem of extending the information on preferences assumed to be known on a subpart of X, to the whole set X, is addressed in section 4, and appears to be the problem of identifying an aggregation model compatible with given intra-criterion and inter-criteria information. In section 5, we show that this problem amounts to define the extension of a given pseudo-Boolean function, and we introduce some possible extensions, which we relate to already known models in the literature (section 6). Section 7 briefly studies the properties of these models, and concludes about their usefulness in this context. In section 8, we show an equivalent set of axioms for our construction, and in section 9, we address the question of unicity of the solution.

This paper does not deal with the practical aspects of the methodology we are proposing, i.e. how to obtain the necessary information for building the aggregation model. However, the MACBETH approach [7] could be most useful for extracting the information from the DM.

Lastly, we want to mention that one of the reasons which have motivated this research is the recent development of multicriteria methods based on capacities and the Choquet integral [2], which seems to open new horizons [12, 18, 20]. In a sense, this paper aims at giving a theoretical foundation of this type of approach in the framework of multicriteria decision making.

2 Basic assumptions

We present two basic assumptions, which are the starting point of our construction. We denote the index set of criteria by $N = \{1, \ldots, n\}$. Considering two acts $x, y \in X$, and $A \subset N$, we will often use the notation (x_A, y_{A^c}) to denote the compound act z where $z_i = x_i$ if $i \in A$ and y_i otherwise. $\wedge, \vee$ denote respectively min and max operators.

2.1 Intra-criterion assumption

We consider the particular subsets $X\rfloor_i$, $i = 1, \ldots, n$, of X, which are defined by:

$$X\rfloor_i = \{(\mathbf{0}_1, \ldots, \mathbf{0}_{i-1}, x_i, \mathbf{0}_{i+1}, \ldots, \mathbf{0}_n) | x_i \in X_i\}.$$

Using our convention, acts in $X\rfloor_i$ are denoted more simply by $(x_i, \mathbf{0}_{\{i\}^c})$.

We assume to have an interval scale denoted v_i on each $X\rfloor_i$, which quantifies the attractiveness for the DM of the elements of $X\rfloor_i$ (assumption A1). In order to simplify the notation, we denote for all $i \in N$, $u_i : X_i \longrightarrow \mathbb{R}$, $x_i \mapsto u_i(x_i) = v_i(x_i, \mathbf{0}_{\{i\}^c})$. Thus, assumption A1 means exactly the following:

(A1.1) $\forall x_i, y_i \in X_i$, $u_i(x_i) \geq u_i(y_i)$ if and only if for the decision maker $(x_i, \mathbf{0}_{\{i\}^c})$ is at least as attractive as $(y_i, \mathbf{0}_{\{i\}^c})$.

(A1.2) $\forall x_i, y_i, z_i, w_i \in X_i$, such that $u_i(x_i) > u_i(y_i)$ and $u_i(w_i) > u_i(z_i)$, we have

$$\frac{u_i(x_i) - u_i(y_i)}{u_i(w_i) - u_i(z_i)} = k, \quad k \in \mathbb{R}^+$$

if and only if the difference of attractiveness that the DM feels between $(x_i, \mathbf{0}_{\{i\}^c})$ and $(y_i, \mathbf{0}_{\{i\}^c})$ is equal to k times the difference of attractiveness between $(w_i, \mathbf{0}_{\{i\}^c})$ and $(z_i, \mathbf{0}_{\{i\}^c})$.

We recognize here information concerning the intra-criterion preferences (i.e. the attractiveness of elements of X_i relatively to C_i), hence the name of the assumption, which is a classical type of information in multicriteria decision aid. Observe however that our presentation avoids the introduction of any independence assumption (preferential or cardinal). This is possible since we have introduced in every set X_i an element $\mathbf{0}_i$ with an absolute meaning in terms of attractiveness. This strong meaning allows us to fix naturally $u_i(\mathbf{0}_i) = 0$,[1] $i = 1, \ldots, n$, and thus to consider u_i as a ratio scale on X_i. We can also take advantage of the remaining degree of freedom to fix the value of $u_i(\mathbf{1}_i)$. Contrarily to the case of $u_i(\mathbf{0}_i)$, no particular value, provided it is positive, is mandatory here. However, since all elements $\mathbf{1}_i$, $i = 1, \ldots, n$ have all the same absolute meaning, we have to choose for $u_i(\mathbf{1}_i)$ the same numerical value for all $i \in \{1, \ldots, n\}$, which implies that the only admissible

[1] which is technically always possible, since an interval scale is defined up to a positive affine transformation $\phi(z) = \alpha z + \beta$, $\alpha > 0$, which means that we have two degrees of freedom.

transformations of the scales u_i, $i \in N$, are of the form $\phi(u_i) = \alpha \cdot u_i$, where $\alpha > 0$ does not depend on i. Thanks to the elements $\mathbf{0}_i$ and $\mathbf{1}_i$, the interval scales u_i become thus *commensurable ratio scales.* In the sequel, we take as a convention $u_i(\mathbf{1}_i) = 1$, for $i = 1, \ldots, n$.

2.2 Inter-criteria assumption

We consider now another subset of X, denoted $X\rceil_{\{0,1\}}$, containing the following elements:

$$X\rceil_{\{0,1\}} := \{(\mathbf{1}_A, \mathbf{0}_{A^c}) | A \subset N\},$$

where $(\mathbf{1}_A, \mathbf{0}_{A^c})$ denotes an act $(x_1, \ldots, x_n)$ with $x_i = \mathbf{1}_i$ if $i \in A$ and $x_i = \mathbf{0}_i$ otherwise, following our convention.

We assume to have an interval scale $u_{\{0,1\}}$ on $X\rceil_{\{0,1\}}$, quantifying the attractiveness for the DM of all elements in this set (assumption A2). This means that:

(A2.1) for all $A, B \subset N$, $u_{\{0,1\}}(\mathbf{1}_A, \mathbf{0}_{A^c}) \geq u_{\{0,1\}}(\mathbf{1}_B, \mathbf{0}_{B^c})$ if and only if for the DM $(\mathbf{1}_A, \mathbf{0}_{A^c})$ is at least as attractive as $(\mathbf{1}_B, \mathbf{0}_{B^c})$.

(A2.2) for all $A, B, C, D \subset N$ such that $u_{\{0,1\}}(\mathbf{1}_A, \mathbf{0}_{A^c}) > u_{\{0,1\}}(\mathbf{1}_B, \mathbf{0}_{B^c})$ and $u_{\{0,1\}}(\mathbf{1}_C, \mathbf{0}_{C^c}) > u_{\{0,1\}}(\mathbf{1}_D, \mathbf{0}_{D^c})$, we have

$$\frac{u_{\{0,1\}}(\mathbf{1}_A, \mathbf{0}_{A^c}) - u_{\{0,1\}}(\mathbf{1}_B, \mathbf{0}_{B^c})}{u_{\{0,1\}}(\mathbf{1}_C, \mathbf{0}_{C^c}) - u_{\{0,1\}}(\mathbf{1}_D, \mathbf{0}_{D^c})} = k, \quad k \in \mathbb{R}^+$$

if and only if the difference of attractiveness felt by the DM between $(\mathbf{1}_A, \mathbf{0}_{A^c})$ and $(\mathbf{1}_B, \mathbf{0}_{B^c})$ is k times the difference of attractiveness between $(\mathbf{1}_C, \mathbf{0}_{C^c})$ and $(\mathbf{1}_D, \mathbf{0}_{D^c})$.

As we did for the case of intra-criterion information, we use the two available degrees of freedom of an interval scale to fix:

$$\begin{aligned} u_{\{0,1\}}(\mathbf{1}_\emptyset, \mathbf{0}_N) &= u_{\{0,1\}}(\mathbf{0}_1, \ldots, \mathbf{0}_n) := 0 \\ u_{\{0,1\}}(\mathbf{1}_N, \mathbf{0}_\emptyset) &= u_{\{0,1\}}(\mathbf{1}_1, \ldots, \mathbf{1}_n) := 1. \end{aligned}$$

Having in mind the meaning of $\mathbf{0}_i$, $i = 1, \ldots, n$, it is natural to impose $u_{\{0,1\}}(\mathbf{0}_1, \ldots, \mathbf{0}_n) = 0$. The scale $u_{\{0,1\}}$ is then a ratio scale. Let us point out that any strictly positive value could have been used instead of 1 for the value of $u_{\{0,1\}}(\mathbf{1}_1, \ldots, \mathbf{1}_n)$. However, it is convenient to impose that the value of $u_{\{0,1\}}(\mathbf{1}_1, \ldots, \mathbf{1}_n)$ is equal to the common value chosen for the $u_i(\mathbf{1}_i)$.

At this point, let us remark that both $u_i(\mathbf{1}_i)$ and $u_{\{0,1\}}(\mathbf{1}_i, \mathbf{0}_{\{i\}^c})$ quantify the attractiveness of act $(\mathbf{1}_i, \mathbf{0}_{\{i\}^c})$ for the DM, however their values are on different ratio scales, but with the same 0 since $u_i(\mathbf{0}_i) = u_{\{0,1\}}(\mathbf{0}_1, \dots, \mathbf{0}_n) = 0$. This means that there exists $K_i > 0$ such that $u_{\{0,1\}}(x_i, \mathbf{0}_{\{i\}^c}) = K_i u_i(x_i)$ for all $x_i \in X_i$. An important consequence of this fact is that, in order to have compatibility between these scales (and hence between assumptions A1 and A2), we must have

$$u_{\{0,1\}}(\mathbf{1}_i, \mathbf{0}_{\{i\}^c}) > u_{\{0,1\}}(\mathbf{0}_1, \dots, \mathbf{0}_n) = 0, \quad \forall i,$$

otherwise no constant K_i could exist. This is not restrictive on a practical point of view as soon as each point of view really corresponds to a concern of the DM.

We suppose in addition that whenever $A \subset B$, the act $(\mathbf{1}_B, \mathbf{0}_{B^c})$ is at least as attractive as $(\mathbf{1}_A, \mathbf{0}_{A^c})$, which is also a natural requirement.

Under these conditions, and introducing the set function $\mu : \mathcal{P}(N) \longrightarrow [0, 1]$ by

$$\mu(A) := u_{\{0,1\}}(\mathbf{1}_A, \mathbf{0}_{A^c}) \tag{1}$$

we have defined a non-additive measure, or *fuzzy measure*, [36] or *capacity* [2], with the additional requirement that $\mu(\{i\}) > 0$. Indeed, a capacity is any non negative set function such that $\mu(\emptyset) = 0$, $\mu(N) = 1$, and $\mu(A) \leq \mu(B)$ whenever $A \subset B$.

3 Interaction among criteria

Except the natural assumptions above for μ (monotonicity and $\mu(i) > 0$ for all $i \in N$), no restriction exists on μ. Let us take 2 criteria to show the range of decision behaviours we can obtain with capacities. We suppose in addition that $\mu(\{1\}) = \mu(\{2\})$, which means that the DM is indifferent between $(\mathbf{1}_1, \mathbf{0}_2)$ and $(\mathbf{0}_1, \mathbf{1}_2)$ (i.e. equal importance of criteria, see section 4), and consider 4 acts x, y, z, t such that (see figure 1):

- $x = (\mathbf{0}_1, \mathbf{0}_2)$
- $y = (\mathbf{0}_1, \mathbf{1}_2)$
- $z = (\mathbf{1}_1, \mathbf{1}_2)$
- $t = (\mathbf{1}_1, \mathbf{0}_2)$

Clearly, z is more attractive than x (written $z \succ x$), but preferences over other pairs may depend on the decision maker. Due to the definition of capacities, we can range from the two extremal following situations (recall that $\mu(\{1,2\}) = 1$ is fixed):

extremal situation 1 (lower bound): we put $\mu(\{1\}) = \mu(\{2\}) = 0$, which is equivalent to the preferences $x \sim y \sim t$, where $\sim$ means indifference (figure 1, left).

extremal situation 2 (upper bound): we put $\mu(\{1\}) = \mu(\{2\}) = 1$, which is equivalent to the preferences $y \sim z \sim t$ (figure 1, middle).

Note that the first bound cannot be reached due to the condition $\mu(i) > 0$. The exact intermediate situation is $\mu(\{1\}) = \mu(\{2\}) = 1/2$, meaning that $z \succ y \sim t \succ x$ (figure 1, right), and the difference of attractiveness between x and y, t respectively is the same than between z and y, t respectively.

The first case corresponds to a situation where the criteria are *complementary*, since both have to be satisfactory in order to get a satisfactory act. Otherwise said, the DM makes a conjunctive aggregation. We say that in such a case, which can be characterized by the fact that $\mu(\{1,2\}) > \mu(\{1\}) + \mu(\{2\})$, there is a *positive interaction* between criteria.

The second case corresponds to a situation where the criteria are *substitutive*, since only one has to be satisfactory in order to get a satisfactory act. Here, the DM aggregates disjunctively. We say that in such a case, which can be characterized by the fact that $\mu(\{1,2\}) < \mu(\{1\}) + \mu(\{2\})$, there is a *negative interaction* between criteria.

In the third case, where we have $\mu(\{1,2\}) = \mu(\{1\}) + \mu(\{2\})$, we say that there is *no interaction* among criteria, they are *non interactive*.

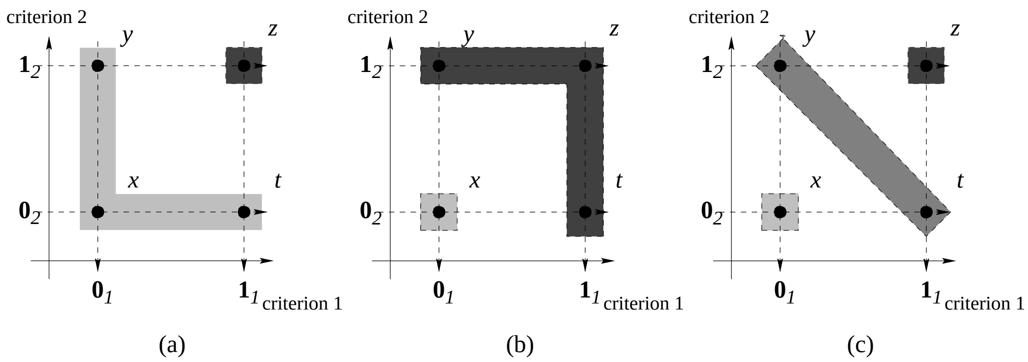

Figure 1: Different cases of interaction

The information we assume to have at hand concerning the attractiveness of acts for the DM is thus perfectly compatible with the interaction situations between criteria, situations which are worth to consider on a practical point of view, but up to now very little studied.

In the above simple example, we had only 2 criteria. In the general case, we use the following definition proposed by Murofushi and Soneda [28].

Definition 1 *The* interaction index *between criteria i and j is given by:*

$$I_{ij} := \sum_{K \subset N \setminus \{i,j\}} \frac{(n-|K|-2)!|K|!}{(n-1)!} [\mu(K \cup \{i,j\}) - \mu(K \cup \{i\}) - \mu(K \cup \{j\}) + \mu(K)]. \quad (2)$$

The definition of this index has been extended to any coalition $A \subset N$ of criteria by Grabisch [14]:

$$I(A) := \sum_{B \subset N \setminus A} \frac{(n-|B|-|A|)!|B|!}{(n-|A|+1)!} \sum_{K \subset A} (-1)^{|A|-|K|} \mu(K \cup B), \forall A \subset N. \quad (3)$$

We have $I_{ij} = I(\{i,j\})$. When $A = \{i\}$, $I(\{i\})$ is nothing else than the Shapley value of game theory [34]. Properties of this set function has been studied and related to the Möbius transform [5]. Also, I has been characterized axiomatically by Grabisch and Roubens [19], in a way similar to the Shapley index. Note that $I_{ij} > 0$ (resp. $< 0, = 0$) for complementary (resp. substitutive, non interactive) criteria.

4 Constructing the model

We will only consider in this paper the general type of aggregation model introduced by Krantz *et al.* [25, Chap. 7]:

> Act $x = (x_1, \ldots, x_n)$ is at least as attractive as act $y = (y_1, \ldots, y_n)$ if and only if
>
> $$F(u_1(x_1), \ldots, u_n(x_n)) \geq F(u_1(y_1), \ldots, u_n(y_n)),$$

where the aggregation function $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ is strictly increasing in all its arguments.

Indeed, this type of model is largely used, and has the advantage of being rather general, and to lead to a complete and transitive preference relation on X.

The central question we deal with in this paper is the identification of an aggregation function F which is compatible with intra-criterion and inter-criteria information defined by assumptions A1 and A2, and satisfies natural conditions. Specifically, we are looking for a mapping $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ of the form

$$F(u_1(x_1), \ldots, u_n(x_n)) = u(x_1, \ldots, x_n)$$

satisfying the following requirements (in which the presence of α is due to the fact that the u_i are commensurable ratio scales):

(i) compatibility with intra-criteria information (assumption A1)

- $\forall i \in N$ and $\forall x_i, y_i \in X_i$,

  $$u_i(x_i) \geq u_i(y_i) \Leftrightarrow u(x_i, \mathbf{0}_{\{i\}^c}) \geq u(y_i, \mathbf{0}_{\{i\}^c})$$

  which becomes, in terms of F (due to the consequences of assumption A1 on the scale):

  $$\begin{aligned} &u_i(x_i) \geq u_i(y_i) \Leftrightarrow \\ &F(0, \ldots, 0, \alpha u_i(x_i), 0, \ldots, 0) \geq F(0, \ldots, 0, \alpha u_i(y_i), 0, \ldots, 0) \end{aligned} \quad (4)$$

  for all $\alpha > 0$. In fact, the constant α here is useless, since for any $\alpha > 0$, $u_i(x_i) \geq u_i(y_i) \Leftrightarrow \alpha u_i(x_i) \geq \alpha u_i(y_i)$.

- $\forall i \in N$ and $\forall w_i, x_i, y_i, z_i$ such that $u_i(w_i) > u_i(x_i)$ and $u_i(y_i) > u_i(z_i)$,

  $$\frac{u(w_i, \mathbf{0}_{\{i\}^c}) - u(x_i, \mathbf{0}_{\{i\}^c})}{u(y_i, \mathbf{0}_{\{i\}^c}) - u(z_i, \mathbf{0}_{\{i\}^c})} = \frac{u_i(w_i) - u_i(x_i)}{u_i(y_i) - u_i(z_i)}$$

  which becomes in terms of F:

  $$\frac{F(0, \ldots, 0, \alpha u_i(w_i), 0, \ldots, 0) - F(0, \ldots, 0, \alpha u_i(x_i), 0, \ldots, 0)}{F(0, \ldots, 0, \alpha u_i(y_i), 0, \ldots, 0) - F(0, \ldots, 0, \alpha u_i(z_i), 0, \ldots, 0)} = \frac{u_i(w_i) - u_i(x_i)}{u_i(y_i) - u_i(z_i)} \quad (5)$$

  for all $\alpha > 0$.

(ii) compatibility with inter-criteria information (assumption A2)

- $\forall A, B \subset N$, we have

$$u_{\{0,1\}}(\mathbf{1}_A, \mathbf{0}_{A^c}) \geq u_{\{0,1\}}(\mathbf{1}_B, \mathbf{0}_{B^c}) \Leftrightarrow u(\mathbf{1}_A, \mathbf{0}_{A^c}) \geq u(\mathbf{1}_B, \mathbf{0}_{B^c})$$

which becomes, in terms of F:

$$u_{\{0,1\}}(\mathbf{1}_A, \mathbf{0}_{A^c}) \geq u_{\{0,1\}}(\mathbf{1}_B, \mathbf{0}_{B^c}) \Leftrightarrow F(\alpha 1_A, 0_{A^c}) \geq F(\alpha 1_B, 0_{B^c})$$

for all $\alpha > 0$, where for any $A \subset N$, $(1_A, 0_{A^c})$ is the vector whose component x_i is 1 whenever $i \in A$, and 0 otherwise.

- $\forall A, B, C, D \subset N$, with $u_{\{0,1\}}(\mathbf{1}_A, \mathbf{0}_{A^c}) > u_{\{0,1\}}(\mathbf{1}_B, \mathbf{0}_{B^c})$ and $u_{\{0,1\}}(\mathbf{1}_C, \mathbf{0}_{C^c}) > u_{\{0,1\}}(\mathbf{1}_D, \mathbf{0}_{D^c})$, we have:

$$\frac{u(\mathbf{1}_A, \mathbf{0}_{A^c}) - u(\mathbf{1}_B, \mathbf{0}_{B^c})}{u(\mathbf{1}_C, \mathbf{0}_{C^c}) - u(\mathbf{1}_D, \mathbf{0}_{D^c})} = \frac{u_{\{0,1\}}(\mathbf{1}_A, \mathbf{0}_{A^c}) - u_{\{0,1\}}(\mathbf{1}_B, \mathbf{0}_{B^c})}{u_{\{0,1\}}(\mathbf{1}_C, \mathbf{0}_{C^c}) - u_{\{0,1\}}(\mathbf{1}_D, \mathbf{0}_{D^c})}$$

which becomes, in terms of F:

$$\frac{F(\alpha 1_A, 0_{A^c}) - F(\alpha 1_B, 0_{B^c})}{F(\alpha 1_C, 0_{C^c}) - F(\alpha 1_D, 0_{D^c})} = \frac{u_{\{0,1\}}(\mathbf{1}_A, \mathbf{0}_{A^c}) - u_{\{0,1\}}(\mathbf{1}_B, \mathbf{0}_{B^c})}{u_{\{0,1\}}(\mathbf{1}_C, \mathbf{0}_{C^c}) - u_{\{0,1\}}(\mathbf{1}_D, \mathbf{0}_{D^c})} \tag{6}$$

for all $\alpha > 0$.

(iii) conditions related to absolute information
We impose that scales u and $u_{\{0,1\}}$ coincide on particular acts corresponding to absolute information, namely:

- $u(\mathbf{0}_1, \dots, \mathbf{0}_n) = u_{\{0,1\}}(\mathbf{0}_1, \dots, \mathbf{0}_n) := 0$, which leads to $F(0, \dots, 0) = 0$.
- $u(\mathbf{1}_1, \dots, \mathbf{1}_n) = u_{\{0,1\}}(\mathbf{1}_1, \dots, \mathbf{1}_n) := 1$, which leads to $F(1, \dots, 1) = 1$. However, remember that the choice of value "1" was arbitrary when building scales u_i and $u_{\{0,1\}}$, and any positive constant α can do. Hence, we should satisfy more generally $F(\alpha, \dots, \alpha) = \alpha$, $\forall \alpha > 0$.

(iv) monotonicity of F. This property is a fundamental requirement for any aggregation function:

$$\forall (t_1, \dots, t_n), \forall (t'_1, \dots, t'_n) \in \mathbb{R}^n,$$
$$t'_i \geq t_i, i = 1, \dots, n \Rightarrow F(t'_1, \dots, t'_n) \geq F(t_n, \dots, t_n).$$

The monotonicity is strict if all inequalities are strict. Remark that monotonicity entails the first condition of **(i)**, namely formula (4).

Let us remark that, as suggested in **(iv)** above, that F can be viewed as an aggregation function, and thus our problem amounts to the search of an aggregation model which is compatible with intra- and inter-criteria information defined by assumptions A1 and A2.

At this point, let us make two remarks.

- the reader may wonder about the very specific form of inter-criteria information asked for, that is, attractiveness of acts of the form $(\mathbf{1}_A, \mathbf{0}_{A^c})$. These acts present the double advantage to be non related with real acts, which permits to avoid any emotional answer from the DM, and to have, taking into account the definition of $\mathbf{0}_i$ and $\mathbf{1}_i$, a very clear meaning, and consequently, to be very well perceived and understood.

  They are currently used in real world applications of the MACBETH approach [6, 8, 9] . Until now, these applications were done in the framework of an additive aggregation model. In such a case, only acts of the form $(\mathbf{1}_i, \mathbf{0}_{\{i\}^c})$ have to be introduced.

  What we are doing here is merely a generalization, considering not only single criteria, but any *coalition* of criteria. This natural generalization from singletons to subsets is indeed the key to the modelling of interaction, as explained in section 3. In this sense, the global utility $u(\mathbf{1}_A, \mathbf{0}_{A^c})$, which is a capacity (see section 2.2), could represent the importance of coalition A to make decision.

- it can be observed that conditions **(ii)** and **(iii)** above entail that the function $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ to be determined must coincide with μ on $\{0,1\}^n$, i.e.:

  $$F(1_A, 0_{A^c}) = \mu(A), \quad \forall A \subset N.$$

  Indeed, just consider equation (6) with $B = D = \emptyset, C = N$, and use **(iii)**, and definition of μ (eq. (1)).

  Thus, F must be an extension of μ on $\mathbb{R}^n$. In other words, *the assignment of importance to coalitions is tightly linked with the evaluation function.* This fact is well known in the MCDM community (see e.g. Mousseau [27]), but the argument above puts it more precisely. The next section addresses in full detail the problem of extending capacities.

5 Extension of pseudo-Boolean functions

The problem of extending a capacity can be nicely formalized through the use of pseudo-Boolean functions (see e.g. [21]).

Any function $f : \{0,1\}^n \longrightarrow \mathbb{R}$ is a said to be a *pseudo-Boolean function*. By making the usual bijection between $\{0,1\}^n$ and $\mathcal{P}(N)$, it is clear that pseudo-Boolean functions on $\{0,1\}^n$ coincide with real-valued set functions on N (of which capacities are a particular case). More specifically, if we define for any subset $A \subset N$ the vector $\delta_A = [\delta_A(1) \cdots \delta_A(n)]$ in $\{0,1\}^n$ by $\delta_A(i) = 1$ if $i \in A$, and 0 otherwise, then for any set function v we can define its associated pseudo-Boolean function f by

$$f(\delta_A) := v(A), \quad \forall A \subset N,$$

and reciprocally. It has been shown by Hammer and Rudeanu [22] that any pseudo-Boolean function can be written in a multilinear form:

$$f(t) = \sum_{A \subset N} m(A) \cdot \prod_{i \in A} t_i, \quad \forall t \in \{0,1\}^n. \tag{7}$$

$m(A)$ corresponds to the Möbius transform (see e.g. Rota [31]) of v, associated to f, which is defined by:

$$m(A) = \sum_{B \subset A} (-1)^{|A \setminus B|} v(B). \tag{8}$$

Reciprocally, v can be recovered from the Möbius transform by

$$v(A) = \sum_{B \subset A} m(B). \tag{9}$$

If necessary, we write m^v for the Möbius transform of v. Note that (7) can be put in an equivalent form, which is

$$f(t) = \sum_{A \subset N} m(A) \cdot \bigwedge_{i \in A} t_i, \quad \forall t \in \{0,1\}^n. \tag{10}$$

More generally, the product can be replaced by any operator $\boxdot$ on $[0,1]^n$ coinciding with the product on $\{0,1\}^n$, such as t-norms [32] (see e.g. [10] for a survey on this topic, and [24] for a complete treatment). We recall that a t-norm is a binary operator T on $[0,1]$ which is commutative, associative, non decreasing in each place, and such that $T(x,1) = x$, for all $x \in [0,1]$. Associativity permits to unambiguously define t-norms for more than 2 arguments.

These are not the only ways to write pseudo-Boolean functions. When v is a capacity, it is possible to replace the sum by $\vee$, as the following formula shows [15]:

$$f(t) = \bigvee_{A \subset N} m_\vee(A) \wedge \left(\bigwedge_{i \in A} t_i \right). \tag{11}$$

The quantity $m_\vee$ is called the *ordinal Möbius transform*, and is related to v by $m_\vee(A) = v(A)$ whenever $v(A) > v(A \setminus i)$ for all $i \in A$, and 0 otherwise. Note that conversely we have (compare with (9)):

$$v(A) = \bigvee_{B \subset A} m_\vee(B), \forall A \subset N. \tag{12}$$

In the sequel, we focus on formulas (7) and (10). We will come back on alternatives to these formulas in section 8.

In order to extend f to $\mathbb{R}^n$, which is necessary in our framework since the DM can judge that an element $(x_i, \mathbf{0}_{\{i\}^c})$ is less attractive than $(\mathbf{0}_1, \ldots, \mathbf{0}_n)$ (in that case $u_i(x_i) < 0$), two immediate extensions come from (7) and (10), where we simply use any $t \in \mathbb{R}^n$ instead of $\{0,1\}^n$. We will denote them

$$f^\Pi(t) := \sum_{A \subset N} m(A) \cdot \prod_{i \in A} t_i, \quad \forall t \in \mathbb{R}^n, \tag{13}$$

$$f^\wedge(t) := \sum_{A \subset N} m(A) \cdot \bigwedge_{i \in A} t_i, \quad \forall t \in \mathbb{R}^n. \tag{14}$$

However a second way can be obtained by considering the fact that any real number t can be written under the form $t = t^+ - t^-$, where $t^+ = t \vee 0$, and $t^- = -t \vee 0$. If, by analogy with this remark, we replace $\prod_i t_i$ by $\prod_i t_i^+ - \prod_i t_i^-$, and similarly with $\bigwedge$, we obtain two new extensions:

$$f^{\Pi\pm}(t) := \sum_{A \subset N} m(A) \left[\prod_{i \in A} t_i^+ - \prod_{i \in A} t_i^- \right], \quad \forall t \in \mathbb{R}^n, \tag{15}$$

$$f^{\wedge\pm}(t) := \sum_{A \subset N} m(A) \left[\bigwedge_{i \in A} t_i^+ - \bigwedge_{i \in A} t_i^- \right], \quad \forall t \in \mathbb{R}^n. \tag{16}$$

These are not the only possible extensions. In fact, nothing prevents us to introduce for the negative part another capacity, e.g. equation (16) could become:

$$f_{12}^{\wedge\pm}(t) := \sum_{A \subset N} m_1(A) \cdot \bigwedge_{i \in A} t_i^+ - \sum_{A \subset N} m_2(A) \cdot \bigwedge_{i \in A} t_i^-, \quad \forall t \in \mathbb{R}^n. \tag{17}$$

However, we will not consider this possibility in the subsequent development, except in section 9 where the question of unicity is addressed. In the next sections we investigate whether extensions (13) to (16) are related to known models of aggregation, and which one satisfy the requirements (i) to (iv) introduced in section 4, and can be thus used as an aggregation function in our case.

6 Link with existing models

We introduce the Choquet integral with respect to a capacity, which has been introduced as an aggregation operator by Grabisch [11, 12]. Let μ be a capacity on N, and $t = (t_1, \ldots, t_n) \in (\mathbb{R}^+)^n$. The *Choquet integral* of t with respect to μ is defined by [29]:

$$\mathcal{C}_\mu(t) = \sum_{i=1}^{n} (t_{(i)} - t_{(i-1)})\mu(\{(i), \ldots, (n)\}) \tag{18}$$

where $\cdot_{(i)}$ indicates a permutation on N so that $t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(n)}$, and $t_{(0)} := 0$ by convention. It can be shown that the Choquet integral can be written as follows:

$$\mathcal{C}_\mu(t) = \sum_{A \subset N} m(A) \bigwedge_{i \in A} t_i, \quad \forall t \in (\mathbb{R}^+)^n \tag{19}$$

where m denotes the Möbius transform of μ. This result has been shown by Chateauneuf and Jaffray [1] (also by Walley [40]), extending Dempster's result [3].

We are now ready to relate previous extensions to known aggregation models.

- the extension f^Π is known in multiattribute utility theory as the *multilinear model* [23], which we denote by MLE. Note that our presentation gives a meaning to the coefficients of the polynom, since they are the Möbius transform of the underlying capacity defined by $\mu(A) = u(\mathbf{1}_A, \mathbf{0}_{A^c})$, for all $A \subset N$. Up to now, no clear interpretation of these coefficients were given.

- concerning $f^{\Pi\pm}$, to our knowledge, it does not correspond to anything known in the literature. We will denote it by SMLE (symmetric MLE).

- considering $f^\wedge$ restricted to $(\mathbb{R}^+)^n$, it appears due to the above result (19) that $f^\wedge$ is the Choquet integral of t with respect to μ, where μ corresponds to f. This extension is also known as the Lovász extension of f [26, 35]. At this point, let us remark that the extension of the Choquet integral to negative arguments has been considered by Denneberg [4], who gives two possibilities:

  1. the symmetric extension $\overset{\mathrm{S}}{\mathcal{C}}_\mu$ defined by

  $$\overset{\mathrm{S}}{\mathcal{C}}_\mu(t) = \mathcal{C}_\mu(t^+) - \mathcal{C}_\mu(t^-), \quad \forall t \in \mathbb{R}^n. \tag{20}$$

2. the asymmetric extension $\overset{\text{AS}}{\mathcal{C}}_\mu$ defined by

$$\overset{\text{AS}}{\mathcal{C}}_\mu(t) = \mathcal{C}_\mu(t^+) - \mathcal{C}_{\bar{\mu}}(t^-), \quad \forall t \in \mathbb{R}^n, \tag{21}$$

where $\bar{\mu}$ is the conjugate capacity defined by $\bar{\mu}(A) := \mu(N) - \mu(A^c)$.

The first extension has been proposed first by Šipoš [39], while the second one is considered as the classical definition of the Choquet integral on real numbers. In the sequel, we will denote the Šipoš integral by $\check{\mathcal{S}}_\mu$, while we keep $\mathcal{C}_\mu$ for the (usual) Choquet integral.

The following proposition gives the expression of Choquet and Šipoš integrals in terms of the Möbius transform, and shows that $f^\wedge \equiv \mathcal{C}_\mu$ and $f^{\wedge\pm} \equiv \check{\mathcal{S}}_\mu$.

Proposition 1 *Let μ be a capacity. For any $t \in \mathbb{R}^n$,*

$$\mathcal{C}_\mu(t) = \sum_{A\subset N} m(A) \bigwedge_{i\in A} t_i, \tag{22}$$

$$\begin{aligned}
\check{\mathcal{S}}_\mu(t) &= \sum_{A\subset N} m(A) \left[\bigwedge_{i\in A} t_i^+ - \bigwedge_{i\in A} t_i^-\right] \\
&= \sum_{A\subset N^+} m(A) \bigwedge_{i\in A} t_i + \sum_{A\subset N^-} m(A) \bigvee_{i\in A} t_i,
\end{aligned} \tag{23}$$

where $N^+ := \{i \in N | t_i \geq 0\}$ and $N^- = N \setminus N^+$.

The proof is based on the following lemma, shown in [16].

Lemma 1 *Let v be any set function such that $v(\emptyset) = 0$, and consider its* co-Möbius *transform*[2] *[13], defined by:*

$$\check{m}^v(A) := \sum_{B\supset N\setminus A} (-1)^{n-|B|} v(B) = \sum_{B\subset A} (-1)^{|B|} v(N \setminus B), \forall A \subset N.$$

Then, if $\bar{v}$ denotes the conjugate set function:

$$\check{m}^{\bar{v}}(A) = (-1)^{|A|+1} m^v(A), \quad \forall A \subset N, A \neq \emptyset \tag{24}$$

and for any $a \in (\mathbb{R}^+)^n$,

$$\mathcal{C}_v(a) = \sum_{A\subset N, A\neq\emptyset} (-1)^{|A|+1} \check{m}^v(A) \bigvee_{i\in A} a_i. \tag{25}$$

[2] Called "commonality function" by Shafer [33].

Proof of Prop. 1: The case of Šipoš integral is clear from (14) and (20). For the case of Choquet, the proof is based on the above lemma. Using (14), we have:

$$\begin{aligned}\mathcal{C}_\mu(t^+) &= \sum_{A\subset N} m(A) \bigwedge_{i\in A} t_i^+ \\ &= \sum_{A\subset N, A\cap N^-=\emptyset} m(A) \bigwedge_{i\in A} t_i\end{aligned}$$

Also, using (24) and (25) and remarking that $m(\emptyset)=0$, we get:

$$\begin{aligned}\mathcal{C}_{\bar{\mu}}(t^-) &= \sum_{A\subset N, A\neq\emptyset} (-1)^{|A|+1}\check{m}^{\bar{\mu}}(A) \bigvee_{i\in A} t_i^- \\ &= \sum_{A\subset N} m(A) \bigvee_{i\in A} t_i^-.\end{aligned}$$

Now

$$\bigvee_{i\in A} t_i^- = \begin{cases} -\bigwedge_{i\in A} t_i, & \text{if } A\cap N^-\neq\emptyset \\ 0, & \text{otherwise}\end{cases}$$

Thus

$$\mathcal{C}_{\bar{\mu}}(t^-) = -\sum_{A\subset N, A\cap N^-\neq\emptyset} m(A) \bigwedge_{i\in A} t_i$$

so that

$$\mathcal{C}_\mu(t) = \mathcal{C}_\mu(t^+) + \mathcal{C}_{\bar{\mu}}(t^-) = \sum_{A\subset N} m(A) \bigwedge_{i\in A} t_i.$$

□

The next proposition gives the expression of Choquet and Šipoš integral directly in terms of the capacity.

Proposition 2 *Let μ be a capacity. For any $t\in\mathbb{R}^n$,*

$$\mathcal{C}_\mu(t) = t_{(1)} + \sum_{i=2}^{n} \left(t_{(i)} - t_{(i-1)}\right)\mu\left(\{(i),\dots,(n)\}\right) \tag{26}$$

$$\begin{aligned}\check{\mathcal{S}}_\mu(t) = &\sum_{i=1}^{p-1} \left(t_{(i)} - t_{(i+1)}\right)\mu\left(\{(1),\dots,(i)\}\right) + t_{(p)}\mu\left(\{(1),\dots,(p)\}\right) \\ &+ t_{(p+1)}\mu\left(\{(p+1),\dots,(n)\}\right) + \sum_{i=p+2}^{n} \left(t_{(i)} - t_{(i-1)}\right)\mu\left(\{(i),\dots,(n)\}\right)\end{aligned} \tag{27}$$

where $\cdot_{(i)}$ *indicates a permutation on* N *so that* $t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(p)} < 0 \leq t_{(p+1)} \leq \cdots \leq t_{(n)}$.

Proof: from the definition (18), we have:

$$\mathcal{C}_\mu(t) = t_{(1)} + \sum_{i=2}^{n} \left(t_{(i)} - t_{(i-1)}\right) \mu\left(\{(i), \dots, (n)\}\right).$$

Let $t \in \mathbb{R}^n$. We split t into its positive and negative parts t^+, t^-. Since

$$\begin{cases} (t^+)_{(1)} = (t^+)_{(2)} = \cdots = (t^+)_{(p)} = 0 \\ (t^+)_{(p+1)} = t_{(p+1)} \\ \vdots \\ (t^+)_{(n)} = t_{(n)} \end{cases}$$

we have

$$\mathcal{C}_\mu(t^+) = t_{(p+1)}\mu\left(\{(p+1), \dots, (n)\}\right) + \sum_{i=p+2}^{n} \left(t_{(i)} - t_{(i-1)}\right) \mu\left(\{(i), \dots, (n)\}\right) .$$

In the same way, one has

$$\mathcal{C}_\mu(t^-) = -t_{(p)}\mu\left(\{(p), \dots, (1)\}\right) - \sum_{i=1}^{p-1} \left(t_{(i)} - t_{(i+1)}\right) \mu\left(\{(i), \dots, (1)\}\right) .$$

This gives the desired expression for Šipoš integral. The case of Choquet integral proceeds similarly. □

Remarking that $\mathcal{C}_\mu(0) = \check{\mathcal{S}}_\mu(0)$ for any capacity, we have from proposition 2:

$$\mathcal{C}_\mu(-t) = -\mathcal{C}_{\bar{\mu}}(t) \tag{28}$$

$$\check{\mathcal{S}}_\mu(-t) = -\check{\mathcal{S}}_\mu(t) \tag{29}$$

for any t in $\mathbb{R}^n$, hence the terms asymmetric and symmetric.

In summary, three among the four extensions correspond to known models of aggregation, even if contexts may differ.

7 Properties of the extensions

This section is devoted to the study of the four extensions, regarding the properties requested in the construction of the aggregation model (section 4).

compatibility with intra-criterion information (assumption A1) Recalling that $u_i(\mathbf{0}_i) = 0\ \forall i \in N$, and noting that $m(\{i\}) = \mu(\{i\})$, a straightforward computation shows that for any $\alpha > 0$:

$$\mathcal{C}_\mu(0, \dots, 0, \alpha u_i(x_i), 0, \dots, 0) = \begin{cases} \alpha\mu(\{i\})u_i(x_i) & \text{if } x_i \succeq_i \mathbf{0}_i \\ \alpha\bar{\mu}(\{i\})u_i(x_i) & \text{if } x_i \prec_i \mathbf{0}_i \end{cases} \tag{30}$$

$$\check{\mathcal{S}}_\mu(0, \dots, 0, \alpha u_i(x_i), 0, \dots, 0) = \alpha\mu(\{i\})u_i(x_i) \tag{31}$$

$$\mathrm{MLE}_\mu(0, \dots, 0, \alpha u_i(x_i), 0, \dots, 0) = \alpha\mu(\{i\})u_i(x_i) \tag{32}$$

$$\mathrm{SMLE}_\mu(0, \dots, 0, \alpha u_i(x_i), 0, \dots, 0) = \alpha\mu(\{i\})u_i(x_i). \tag{33}$$

In the general case, we have $\mu(\{x_i\}) \neq \bar{\mu}(\{x_i\})$. Thus there is an angular point around the origin for the Choquet integral. The consequence is that equation (5), and hence assumption A1, are not satisfied by the Choquet integral in general.

This curious property can be explained as follows. For the Šipoš integral, the zero has a special role, since it is the zero of the ratio scale, and all is symmetric with respect to this point. For the Choquet integral, the zero has no special meaning, but observe that if $x_i \succeq \mathbf{0}_i \succeq y_i$, the acts $(\mathbf{0}_1, \dots \mathbf{0}_{i-1}, x_i, \mathbf{0}_{i+1}, \dots, \mathbf{0}_n)$ and $(\mathbf{0}_1, \dots \mathbf{0}_{i-1}, y_i, \mathbf{0}_{i+1}, \dots, \mathbf{0}_n)$ are not comonotonic, i.e. they induce a different ordering of the integrand.

compatibility with inter-criteria information (assumption A2) It results from the definitions of $\mathcal{C}_\mu$, $\check{\mathcal{S}}_\mu$, MLE_μ and SMLE_μ that, $\forall A \subset N$ and $\forall \alpha > 0$,

$$\mathrm{MLE}_\mu(\alpha 1_A, 0_{A^c}) = \mathrm{SMLE}_\mu(\alpha 1_A, 0_{A^c}) = \sum_{B \subset A} m(B)\alpha^{|B|}, \tag{34}$$

and

$$\mathcal{C}_\mu(\alpha 1_A, 0_{A^c}) = \check{\mathcal{S}}_\mu(\alpha 1_A, 0_{A^c}) = \alpha\mu(A).$$

Consequently, MLE and SMLE are inadequate for our model.

use of absolute information Obviously any extension satisfies $F(0, \dots, 0) = 0$, and taking into account the fact that $\mu(N) = 1$, we have $\mathcal{C}_\mu(\alpha, \dots, \alpha) = \check{\mathcal{S}}_\mu(\alpha, \dots, \alpha) = \alpha$, for all $\alpha > 0$. But from (34), this property is not satisfied by MLE and SMLE.

Monotonicity It can be shown that, for any $t, t' \in \mathbb{R}^n$,

$$t_i \leq t'_i, i = 1, \dots, n \Rightarrow \mathcal{C}_\mu(t_1, \dots, t_n) \leq \mathcal{C}_\mu(t'_1, \dots, t'_n) \tag{35}$$

$$t_i \leq t'_i, i = 1, \dots, n \Rightarrow \check{\mathcal{S}}_\mu(t_1, \dots, t_n) \leq \check{\mathcal{S}}_\mu(t'_1, \dots, t'_n). \tag{36}$$

This well-known result (see e.g. Denneberg [4]) comes from the fact that for any $t \in (\mathbb{R}^+)^n$, an equivalent form of (18) is:

$$\mathcal{C}_\mu(t) = \sum_{i=1}^n t_{(i)}[\mu(\{(i), \ldots, (n)\}) - \mu(\{(i+1), \ldots, (n)\})].$$

Monotonicity is immediate from the fact that $A \subset B$ implies $\mu(A) \leq \mu(B)$. Now, for any $t \in \mathbb{R}^n$, monotonicity of the Choquet and Šipoš integrals follow from equations (20) and (21). To obtain strict monotonicity, we need strict monotonicity of the capacity, i.e. $A \subsetneq B$ implies $\mu(A) < \mu(B)$.

It is easy to see from definition that MLE and SMLE are monotonic when the coefficients $m(A)$ are all positive. But in general, the Möbius transform of a capacity is not always positive. To our knowledge, there is no result in the general case. The following can be proven.

Proposition 3 *For any $t \in [0,1]^n$, for any capacity μ, MLE_μ is non decreasing with respect to t_i, $i = 1, \ldots, n$. Strict increasingness is ensured iff μ is strictly monotonic.*

Proof: We can express easily MLE with respect to μ (see Owen [30]):

$$\mathrm{MLE}_\mu(t) = \sum_{A \subset N} \left[\prod_{i \in A} t_i\right] \left[\prod_{i \notin A} (1 - t_i)\right] \mu(A).$$

Then we have, for any $t \in [0,1]^n$ and any $k \in N$:

$$\begin{aligned}
\frac{\partial \mathrm{MLE}(t)}{\partial t_k} &= \sum_{A \subset N \setminus k} \left[\prod_{i \in A} t_i\right] \left[\prod_{i \notin A, i \neq k} (1 - t_i)\right] \mu(A \cup k) \\
&\quad - \sum_{A \subset N \setminus k} \left[\prod_{i \in A} t_i\right] \left[\prod_{i \notin A, i \neq k} (1 - t_i)\right] \mu(A) \\
&= \sum_{A \subset N \setminus k} \left[\prod_{i \in A} t_i\right] \left[\prod_{i \notin A, i \neq k} (1 - t_i)\right] (\mu(A \cup k) - \mu(A)).
\end{aligned}$$

Clearly, the expression is non negative (resp. positive) for any $k \in N$ iff μ is monotonic (resp. strictly monotonic). $\square$

The proof shows clearly that MLE could be non increasing when t is no more in $[0,1]^n$. Taking for example $n = 2$, with $\mu(\{1\}) = \mu(\{2\}) = 0.9$, we have:

$$\begin{aligned}
&\mathrm{MLE}_\mu(1,1) = 0.9 + 0.9 - 0.8 = 1 \\
&\mathrm{MLE}_\mu(3,3) = (3)(0.9) + (3)(0.9) - (9)(0.8) = -1.8 < \mathrm{MLE}_\mu(1,1).
\end{aligned}$$

As a consequence, the use of MLE should be restricted to criteria of which scores are limited to $[0, 1]$, that is, *unipolar bounded* criteria. Also, SMLE which differs from MLE only for negative values, is clearly useless.

Scale preservation Although this property is not required by our construction (but it somehow underlies it in assumptions A1 and A2), it is interesting to investigate whether the extensions satisfy it.

The following is easy to prove.

(C.1) invariance to the same positive affine transformation

$$\mathcal{C}_\mu(\alpha t_1 + \beta, \ldots, \alpha t_n + \beta) = \alpha \mathcal{C}_\mu(t_1, \ldots, t_n) + \beta, \quad \forall \alpha \geq 0, \forall \beta \in \mathbb{R}.$$

(S.1) homogeneity

$$\check{\mathcal{S}}_\mu(\alpha t_1, \ldots, \alpha t_n) = \alpha \check{\mathcal{S}}_\mu(t_1, \ldots, t_n), \forall \alpha \in \mathbb{R}.$$

As remarked by Sugeno and Murofushi [37], this means that if the scores t_i are on commensurable interval scales, then the global score computed by the Choquet integral is also on an interval scale (i.e. relative position of the zero), and if the scores are on a ratio scale, then the global score computed by the Šipoš integral is on a ratio scale (absolute position of the zero).

By contrast, MLE and SMLE neither preserve the interval nor the ratio scale, since they are not homogeneous. Indeed, taking $n = 2$ and any $\alpha \in \mathbb{R}^*$:

$$\begin{aligned} \mathrm{MLE}_\mu(\alpha t_1, \alpha t_2) &= m(\{1\})\alpha t_1 + m(\{2\})\alpha t_2 + m(\{1,2\})\alpha^2 t_1 t_2 \\ &\neq \alpha \mathrm{MLE}_\mu(t_1, t_2). \end{aligned}$$

This is the reason why MLE and SMLE failed to fulfill assumption A2. Note however that MLE satisfies (5) but not (6).

As a conclusion, only the Šipoš integal among our four candidates can fit all requirements of our construction.

8 An equivalent axiomatic

Our construction is based on a certain number of requirements for aggregation function F, which we sum up below:

- restricted monotonicity (M1), coming from assumption A1:

$$\forall i = 1, \ldots, n, \forall a_i, a_i' \in \mathbb{R}, a_i \geq a_i' \Rightarrow F(a_i, 0_{\{i\}^c}) \geq F(a_i', 0_{\{i\}^c})$$

- interval scale for intra-criterion information (A1):

$$\frac{F(\alpha a_i, 0_{\{i\}^c}) - F(\alpha b_i, 0_{\{i\}^c})}{F(\alpha c_i, 0_{\{i\}^c}) - F(\alpha d_i, 0_{\{i\}^c})} = \frac{a_i - b_i}{c_i - d_i}, \forall \alpha > 0, \forall a_i, b_i, c_i, d_i \in \mathbb{R}, c_i \neq d_i$$

- interval scale for inter-criteria information (A2):

$$\frac{F(\alpha 1_A, 0_{A^c}) - F(\alpha 1_B, 0_{B^c})}{F(\alpha 1_C, 0_{C^c}) - F(\alpha 1_D, 0_{D^c})} = \frac{\mu(A) - \mu(B)}{\mu(C) - \mu(D)}, \quad \forall \alpha > 0$$

- idempotence (I):

$$F(\alpha, \dots, \alpha) = \alpha, \quad \forall \alpha \geq 0,$$

  with restricted versions (I0) for $\alpha = 0$ and (I1) for $\alpha = 1$.

- monotonicity (M), which is non decreasingness of F for each place.

As already noted, (M) implies (M1). All these requirements come from considerations linked with the preference of the DM and scales of measurement. It is possible to show that they are equivalent to a much simpler set of axioms about F.

Proposition 4 *Let $F : \mathbb{R}^n \Rightarrow \mathbb{R}$ and μ a capacity on N. Then the set of axioms (A1), (A2), (I), (M) is equivalent to the following set of axioms:*

1. *homogeneous extension (HE):*

$$F(\alpha 1_A, 0_{A^c}) = \alpha\mu(A), \quad \forall \alpha \geq 0, \forall A \subset N$$

2. *restricted affinity (A)*

$$F(a_i, 0_{\{i\}^c}) = a_i F(1_i, 0_{\{i\}^c}), \quad \forall a_i \in \mathbb{R}, \forall i = 1, \dots, n$$

3. *monotonicity (M).*

Proof: ($\Rightarrow$) Letting $B = D = \emptyset, C = N$ in (A2) and using (I) lead to $F(\alpha 1_A, 0) = \alpha\mu(A)$, which is (HE). Now, using (A1) with $b_i = d_i = 0$, $c_i = 1$, $\alpha = 1$ and using (I0) we get $F(a_i, 0_{\{i\}^c}) = a_i F(1_i, 0_{\{i\}^c})$, which is (A).

($\Leftarrow$) Using (A), we get:

$$\begin{aligned}\frac{F(\alpha a_i, 0_{\{i\}^c}) - F(\alpha b_i, 0_{\{i\}^c})}{F(\alpha c_i, 0_{\{i\}^c}) - F(\alpha d_i, 0_{\{i\}^c})} &= \frac{\alpha a_i F(1_i, 0_{\{i\}^c}) - \alpha b_i F(1_i, 0_{\{i\}^c})}{\alpha c_i F(1_i, 0_{\{i\}^c}) - \alpha d_i F(1_i, 0_{\{i\}^c})} \\ &= \frac{a_i - b_i}{c_i - d_i},\end{aligned}$$

which proves (A1). Now, from (HE) we get immediately

$$\frac{F(\alpha 1_A, 0_{A^c}) - F(\alpha 1_B, 0_{B^c})}{F(\alpha 1_C, 0_{C^c}) - F(\alpha 1_D, 0_{D^c})} = \frac{\mu(A) - \mu(B)}{\mu(C) - \mu(D)}$$

which is (A2). Finally, from (HE) with $A = N$, we get (I) since $\mu(N) = 1$.
$\square$

Nota: (M) can be dropped from the 2 sets of axioms without changing the equivalence.

9 The unicity issue

Having this simpler set of axioms, we address the question of the unicity of the solution, i.e. is the Šipoš integral the only aggregation function satisfying the requirements?

First we examine the following extension on $[0,1]^n$ of pseudo-Boolean functions:

$$F(a_1, \dots, a_n) = \sum_{A \subset N} m(A) \cdot (\underset{i \in A}{\boxdot} a_i), \forall a_i \in [0,1] \tag{37}$$

as suggested in section 5, where $\boxdot$ is a "pseudo-product". Recall that m is the Möbius transform of the underlying capacity. Let us suppose as a basic requirement that $\boxdot$ is a commutative and associative operator, otherwise our expression of F would be ill-defined since $\boxdot_{i \in A} a_i$ would depend on the order of elements in A (commutativity), and on the grouping of elements (associativity). Thus, it is sufficient to define $\boxdot$ on $[0,1]^2$. The following can be shown.

Proposition 5 *Let* $\boxdot : [0,1]^2 \longrightarrow [0,1]$ *be a commutative and associative operator, and F be given by (37). Then:*

(i) F satisfies (HE) on $[0,1]^n$ if and only if $\boxdot$ coincide with the product on $\{0,1\}$, satisfies $\alpha \boxdot \alpha = \alpha$ for all $\alpha \in [0,1]$, and $\alpha \boxdot 0 = 0$.

(ii) F satisfies (M) implies $\boxdot$ is non decreasing.

Proof: (i) ($\Rightarrow$) Let us consider the particular capacity $u_{1,2}$ defined by $u_{1,2}(A) = 1$ if $\{1,2\} \subset A$, and 0 otherwise (unanimity game). It is easy to see that its Möbius transform is such that $m(\{1,2\}) = 1$ and 0 elsewhere. Let us consider (HE) with $A = \emptyset$, $\alpha = 1$, and the capacity $u_{1,2}$. We obtain

$$F(0, \dots, 0) = 1 \cdot (0 \boxdot 0) = u_{1,2}(\emptyset) = 0,$$

hence $0 \boxdot 0 = 0$. Taking now $A = N$, we get:

$$F(1, \dots, 1) = 1 \cdot (1 \boxdot 1) = u_{1,2}(N) = 1,$$

hence $1 \boxdot 1 = 1$. Now let us take $A = \{1\}$, with any $\alpha > 0$ and we obtain from (HE):

$$F(\alpha, 0, \dots, 0) = 1 \cdot (\alpha \boxdot 0) = \alpha u_{1,2}(\{1\}) = 0,$$

hence $\alpha \boxdot 0 = 0$ for any $= \alpha > 0$, in particular when $\alpha = 1$. Thus, $\boxdot$ coincides with the product on $\{0, 1\}$. Lastly, let us apply (HE) with $A = N$ and again the capacity $u_{1,2}$. We obtain:

$$F(\alpha, \alpha, \dots, \alpha) = 1 \cdot (\alpha \boxdot \alpha) = \alpha$$

hence $\alpha \boxdot \alpha = \alpha$.

($\Leftarrow$) For any capacity μ, any $A \subset N$, any $\alpha \in [0, 1]$:

$$\begin{aligned} F(\alpha 1_A, 0_{A^c}) &= \sum_{B \subset A} m(B) \cdot (\mathop{\boxdot}_{i \in B} \alpha) + \sum_{B \not\subset A} m(B) \cdot [(\mathop{\boxdot}_{i \in A} \alpha) \boxdot (\mathop{\boxdot}_{i \notin A} 0)] \\ &= \alpha \sum_{B \subset A} m(B) + 0 \\ &= \alpha \mu(A). \end{aligned}$$

(ii) If $\boxdot$ is decreasing in some place, and m is positive, then F cannot be increasing, a contradiction. Thus, $\boxdot$ is non decreasing in each place. $\square$

To go further in the analysis, let us assume in the sequel that $\boxdot$ is non decreasing. Then we obtain the following result.

Corollary 1 *Let $\boxdot : [0, 1]^2 \longrightarrow [0, 1]$ be a commutative, associative, and non decreasing operator, and F be given by (37). The following propositions are equivalent:*

(i) F satisfies (HE), (M) and (A) on $[0, 1]^n$.

(ii) $\boxdot$ coincide with the product on $\{0, 1\}$, and satisfies $\alpha \boxdot \alpha = \alpha$ for all $\alpha \in [0, 1]$.

Proof: clear from Prop. 5, the fact that (A) is implied by (HE) when working on positive numbers, and the fact that $\alpha \boxdot 0 = 0$ is implied by $0 \boxdot 0 = 0 = 1 \boxdot 0$ and non decreasingness. $\square$

This result gives necessary and sufficient conditions for $\boxdot$ in order to be consistent with our construction.

Adding the requirement $1 \boxdot \alpha = \alpha$ for all $\alpha \in [0,1]$, operator $\boxdot$ becomes a t-norm, as defined in Section 5. Then, the only solution to this set of requirements is the minimum operator [24]. Indeed, taking $\alpha, \beta \in [0,1]$ such that $\alpha \leq \beta$, we have $\alpha = \alpha \boxdot \alpha \leq \beta \boxdot \alpha \leq 1 \boxdot \alpha = \alpha$. This means that the Šipoš integral (for numbers in $[0,1]$, hence it is the Choquet integral) is the only solution with this form of pseudo-Boolean function. However, without this additional assumption, other solutions may exist.

Interestingly enough, the requirement $1 \boxdot \alpha = \alpha$ has a clear interpretation in terms of F. Indeed, for any $A \subset N$, and any $\alpha \in [0,1]$,

$$\begin{aligned}
F(1_A, \alpha_{A^c}) &= \sum_{B \subset A} m(B).1 + \sum_{B \not\subset A} m(B).\alpha \\
&= \sum_{B \subset A} m(B) + \alpha(1 - \sum_{B \subset A} m(B)) \\
&= \alpha + (1-\alpha)\mu(A) \\
&= \alpha + F((1-\alpha)1_A, 0_{A^c}).
\end{aligned}$$

This last expression shows an additivity property of F with particular acts, specifically:

$$F(1_A, \alpha_{A^c}) = F((1-\alpha)1_A, 0_{A^c}) + F(\alpha, \ldots, \alpha).$$

It also shows that F induces a difference scale for those acts, since the zero can be shifted and set to α without any change.

We now present a solution in the spirit of equation (11), which is in fact the Sugeno integral [36] (see [15]). Let us first restrict to positive numbers. We introduce the following aggregation function on $\mathbb{R}^+$:

$$S_{m_\vee}(a_1, \ldots, a_n) = \bigvee_{B \subset N} \Big[m_\vee(B) \cdot \bigwedge_{i \in B} a_i \Big]. \tag{38}$$

This is a variant of Sugeno integral where the product takes place of the minimum operator, which satisfies all requirements when restricted to $\mathbb{R}^+$:

- monotonicity (M): clear since $m_\vee$ is a non negative set function.
- (HE): using equation (12) we get:

$$S_{m_\vee}(\alpha 1_A, 0_{A^c}) = \bigvee_{B \subset A} m_\vee(B) \cdot \alpha = \alpha \cdot \mu(A) = \alpha S_{m_\vee}(1_A, 0_{A^c}).$$

- (A) for positive numbers is simply a particular case of (HE).

Note that (HE) works thanks to the product operator in $S_{m_\vee}$. Thus the original Sugeno integral would not work.

We have to extend this definition for negative numbers in a way similar to the Šipoš integral. The problem of extending the Sugeno integral on negative numbers has been studied by Grabisch [17], in an ordinal framework. We adapt this approach to our case and propose the following:

$$S_{m_\vee}(a_1, \dots, a_n) = S_{m_\vee}(a_1^+, \dots, a_n^+) \circledvee (-S_{m_\vee}(a_1^-, \dots, a_n^-)) \tag{39}$$

with usual notations, and $\circledvee$ (called *symmetric maximum*) is defined by:

$$a \circledvee b = \begin{cases} a, & \text{if } |a| > |b| \\ 0, & \text{if } b = -a \\ b, & \text{otherwise.} \end{cases}$$

The main properties of the symmetric maximum are $a \circledvee 0 = a$ for all $a \in \mathbb{R}$ (existence of a unique neutral element), and $a \circledvee (-a) = 0$ for all $a \in \mathbb{R}$ (existence of a unique symmetric element). Also, it is non decreasing in each place, and associative on $\mathbb{R}^+$ and $\mathbb{R}^-$.

It suffices to verify that (M) and (A) still hold. (M) comes from non decreasingness of $\circledvee$ and $S_{m_\vee}$ for positive arguments. Let us consider $a_i < 0$. Then

$$S_{m_\vee}(a_i, 0_{\{i\}^c}) = 0 \circledvee (-a_i^- S_{m_\vee}(1_i, 0_{\{i\}^c})) = a_i S_{m_\vee}(1_i, 0_{\{i\}^c}).$$

Thus the proposed $S_{m_\vee}$ satisfies all requirements of our construction.

Let us examine now a third way to find other solutions. It was suggested in Section 5, formula (17), which we reproduce here with suitable notations:

$$F(a_1, \dots, a_n) = \sum_{A \subset N} m_1(A) \cdot \bigwedge_{i \in A} a_i^+ - \sum_{A \subset N} m_2(A) \cdot \bigwedge_{i \in A} a_i^-, \quad \forall a \in \mathbb{R}^n.$$

with $a_i^+ := a_i \vee 0$ and $a_i^- = -a_i \vee 0$. This aggregation function is built from two different capacities μ_1, μ_2, one for positive numbers, and the other one for negative numbers. On each part, it is a Choquet integral. Let us mention here that this type of function is well-known in Cumulative Prospect Theory [38]. Obviously, F satisfies (M) and (HE), let us check (A) for negative numbers. We have for any $i \in N$, any $a_i < 0$:

$$F(a_i, 0_{\{i\}^c}) = 0 - m_2(\{i\}) a_i^- = a_i m_2(\{i\}).$$

But $F(1_i, 0_{\{i\}^c}) = m_1(\{i\})$, so that a necessary and sufficient condition to ensure the compatibility with our construction is:

$$m_2(\{i\}) = m_1(\{i\}), \quad \forall i \in N.$$

At this stage, we do not know if other solutions exist, and a complete characterization is left for further study.

10 Conclusion

We have shown in this paper that considering, besides classical comparative information, absolute information, strongly modifies the aggregation problem in MCDA. The classical multilinear model is no more adequate but new models like Choquet and Šipoš integrals appear because absolute information allows to lead to commensurable scales. Among these two models, we have shown that the Šipoš integral is the only acceptable solution, although there exist other models fitting all the requirements. The approach leading to the unicity of the solution based on Šipoš integral is deserved for a subsequent study.

References

[1] A. Chateauneuf and J.Y. Jaffray. Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion. *Mathematical Social Sciences*, 17:263–283, 1989.

[2] G. Choquet. Theory of capacities. *Annales de l'Institut Fourier*, 5:131–295, 1953.

[3] A.P. Dempster. Upper and lower probabilities induced by a multivalued mapping. *Ann. Math. Statist.*, 38:325–339, 1967.

[4] D. Denneberg. *Non-Additive Measure and Integral.* Kluwer Academic, 1994.

[5] D. Denneberg and M. Grabisch. Interaction transform of set functions over a finite set. *Information Sciences*, 121:149–170, 1999.

[6] C.A. Bana e Costa, L. Ensslin, E.C. Correa, and J.C. Vansnick. Decision support systems in action: integrated application in a multicriteria decision aid process. *European J. of Operational Research*, 113:315–335, 1999.

[7] C.A. Bana e Costa and J.C. Vansnick. A theoretical framework for Measuring Attractiveness by a Categorical Based Evaluation TecHnique (MACBETH). In *Proc. XIth Int. Conf. on MultiCriteria Decision Making*, pages 15–24, Coimbra, Portugal, August 1994.

[8] C.A. Bana e Costa and J.C. Vansnick. Applications of the MACBETH approach in the framework of an additive aggregation model. *J. of Multicriteria Decision Analysis*, 6:107–114, 1997.

[9] C.A. Bana e Costa and J.C. Vansnick. The MACBETH approach: basic ideas, software and an application. In N. Meskens and M. Roubens, editors, *Advances in Decision Analysis*, pages 131–157. Kluwer Academic Publishers, 1999.

[10] J.C. Fodor and M. Roubens. *Fuzzy Preference Modelling and Multi-Criteria Decision Aid.* Kluwer Academic Publisher, 1994.

[11] M. Grabisch. Fuzzy integral in multicriteria decision making. *Fuzzy Sets & Systems*, 69:279–298, 1995.

[12] M. Grabisch. The application of fuzzy integrals in multicriteria decision making. *European J. of Operational Research*, 89:445–456, 1996.

[13] M. Grabisch. Alternative representations of discrete fuzzy measures for decision making. *Int. J. of Uncertainty, Fuzziness, and Knowledge Based Systems*, 5:587–607, 1997.

[14] M. Grabisch. k-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, 92:167–189, 1997.

[15] M. Grabisch. On the representation of k-decomposable measures. In *7th IFSA World Congress*, Prague, Czech Republic, June 1997.

[16] M. Grabisch. The interaction and Möbius representations of fuzzy measures on finite spaces, k-additive measures: a survey. In M. Grabisch, T. Murofushi, and M. Sugeno, editors, *Fuzzy Measures and Integrals — Theory and Applications*, pages 70–93. Physica Verlag, 2000.

[17] M. Grabisch. Symmetric and asymmetric fuzzy integrals: the ordinal case. In *6th Int. Conf. on Soft Computing (Iizuka'2000)*, Iizuka, Japan, October 2000.

[18] M. Grabisch, J.M. Baret, and M. Larnicol. Analysis of interaction between criteria by fuzzy measure and its application to cosmetics. In *Int. Conf. on Methods and Applications of Multicriteria Decision Making*, pages 22–25, Mons, Belgium, May 1997.

[19] M. Grabisch and M. Roubens. An axiomatic approach to the concept of interaction among players in cooperative games. *Int. Journal of Game Theory*, 28:547–565, 1999.

[20] M. Grabisch and M. Roubens. Application of the Choquet integral in multicriteria decision making. In M. Grabisch, T. Murofushi, and M. Sugeno, editors, *Fuzzy Measures and Integrals — Theory and Applications*, pages 348–374. Physica Verlag, 2000.

[21] P.L. Hammer and R. Holzman. On approximations of pseudo-Boolean functions. *ZOR - Methods and Models of Operations Research*, 36:3–21, 1992.

[22] P.L. Hammer and S. Rudeanu. *Boolean Methods in Operations Research and Related Areas.* Springer, 1968.

[23] R.L. Keeney and H. Raiffa. *Decision with Multiple Objectives.* Wiley, New York, 1976.

[24] E.P. Klement, R. Mesiar, and E. Pap. *Triangular Norms.* Kluwer Academic Publishers, Dordrecht, 2000.

[25] D.H. Krantz, R.D. Luce, P. Suppes, and A. Tversky. *Foundations of measurement*, volume 1: Additive and Polynomial Representations. Academic Press, 1971.

[26] L. Lovász. Submodular function and convexity. In A. Bachem, M. Grötschel, and B. Korte, editors, *Mathematical programming. The state of the art*, pages 235–257. Springer Verlag, 1983.

[27] V. Mousseau. Analyse et classification de la littérature traitant de l'importance relative des critères en aide multicritère à la décision. *Recherche Opérationnelle/Operations Research*, 26(4):367–389, 1992.

[28] T. Murofushi and S. Soneda. Techniques for reading fuzzy measures (III): interaction index. In *9th Fuzzy System Symposium*, pages 693–696, Sapporo, Japan, May 1993. In Japanese.

[29] T. Murofushi and M. Sugeno. An interpretation of fuzzy measure and the Choquet integral as an integral with respect to a fuzzy measure. *Fuzzy Sets & Systems*, 29:201–227, 1989.

[30] G. Owen. Multilinear extensions of games. In A.E. Roth, editor, *The Shapley Value. Essays in Honor of Lloyd S. Shapley*, pages 139–151. Cambridge University Press, 1988.

[31] G.C. Rota. On the foundations of combinatorial theory I. Theory of Möbius functions. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 2:340–368, 1964.

[32] B. Schweizer and A. Sklar. *Probabilistic Metric Spaces.* North-Holland, New York, 1983.

[33] G. Shafer. *A Mathematical Theory of Evidence.* Princeton Univ. Press, 1976.

[34] L.S. Shapley. A value for n-person games. In H.W. Kuhn and A.W. Tucker, editors, *Contributions to the Theory of Games, Vol. II*, number 28 in Annals of Mathematics Studies, pages 307–317. Princeton University Press, 1953.

[35] I. Singer. Extensions of functions of 0-1 variables and applications to combinatorial optimization. *Numerical Functional Analysis and Optimization*, 7(1):23–62, 1984.

[36] M. Sugeno. *Theory of fuzzy integrals and its applications*. PhD thesis, Tokyo Institute of Technology, 1974.

[37] M. Sugeno and T. Murofushi. *Fuzzy measure theory*, volume 3 of *Course on fuzzy theory*. Nikkan Kōgyō, 1993. In Japanese.

[38] A. Tversky and D. Kahneman. Advances in prospect theory: cumulative representation of uncertainty. *J. of Risk and Uncertainty*, 1992.

[39] J. Šipoš. Integral with respect to a pre-measure. *Math. Slovaca*, 29:141–155, 1979.

[40] P. Walley. Coherent lower (and upper) probabilities. Technical Report 22, University of Warvick, Coventry, 1981.