COMPLEXITY CLASSIFICATION OF OPERATORS IN REGULAR REWRITING SYSTEMS

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ABSTRACT. Regular tree rewriting systems identify with a representation matrix from which we make an algebraic analysis in order to evaluate the average complexity of all the system operators. Then, a joint analysis of the dominating eigenvalue and the directed graph of the representation matrix gives a classification of the system operators using the results of singularity analysis as mathematical background.

CLASSIFICATION DES OPÉRATEURS PAR COMPLEXITÉ DANS UN SYSTÈME DE RÉÉCRITURE RÉGULIER

RÉSUMÉ. Un système de réécriture régulier est caractérisé par une matrice de représentation dont l'analyse algébrique permet d'évaluer la complexité moyenne des opérateurs du système. L'analyse de la valeur propre dominante de la matrice de représentation et de son graphe orienté dégage une classification des opérateurs du système en s'appuyant sur des résultats d'analyse de singularité.

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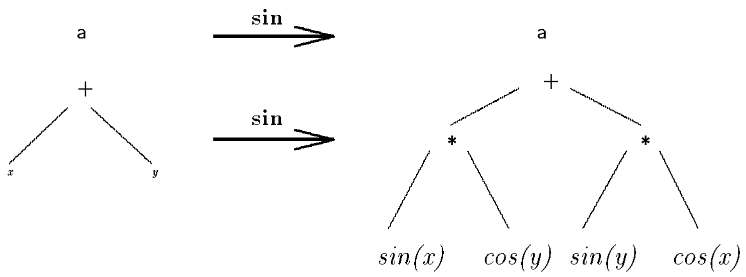
1. INTRODUCTION

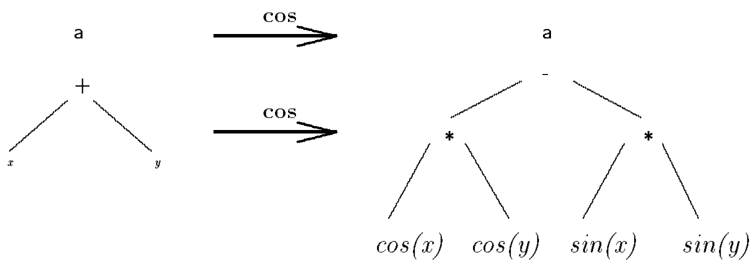
This paper studies regular tree rewriting systems according to the work of C. Choppy, S. Kaplan and M. Soria [2].

An algebraic specification can be expressed as a term tree constructed with basic constructors and operators defined by rewriting rules. The cost of an operator is the number of steps necessary to reduce a term tree to its normal form (we know from [2] that this reduction is confluent and the cost is independent from the rewriting strategy). Such rewriting systems are typically made of expanding rules that recursively create tree duplications and contracting rules that stop the process. We are interested in the global behavior of rewriting systems and will analyze the average cost of the rewriting operators.

Here are two examples to give an idea of what is going to be discussed:

Example 1. This is a formal expansion of operators *cos* and *sin* computing over binary term trees built out of binary constructor $+$ and constant a.





The average complexity — expressed in terms of the average size of the resulting tree — of both operators *sin* and *cos* is equivalent to $3/32\sqrt{2\pi}(4/\sqrt{3})^n n^{3/2}$, n odd, where n is the number of nodes of the initial tree. ♦

Example 2. We are treating a formal differentiation with term copy that computes over binary trees whose nodes are multiplication $*$ and constant a. The rewriting rules form the following system:

$$\begin{aligned} d(a) &\longrightarrow a \\ d(*(u,v)) &\longrightarrow +(*(d(u),cp(v)),*(d(v),cp(u))) \end{aligned}$$

$$\begin{aligned} cp(a) &\longrightarrow a \\ cp(*(u,v)) &\longrightarrow *(cp(u),cp(v)) \end{aligned}$$

The average complexity of the derivative operator is of order $n^{3/2}$ while the copy operator is linear. ♦

The cost of an operator is evaluated through its generating function. We shall see that the operator cost generating functions of a rewriting system are linearly dependent. Thus, all the information about the rewriting system is contained in a *representation matrix* from which arise nice properties. Indeed, to analyze the dominant singularity of cost generating functions, we use the nonnegativity property of representation matrices through Perron-Frobenius theory.

Since the cost generating function of an operator is a quotient of determinants, we manage to get the representation matrix into an irreducible block triangular form whenever it is possible. Then we study each block dominating eigenvalue and deduce the asymptotic behavior (constant, polynomial, exponential) of the average cost of that operator.

Sections 2 and 3 are devoted to recapitulate essential results of [2]. Section 4 will discuss the structure of representation matrices and state equivalences to determine the singularity of smallest module using the dominating eigenvalue of irreducible matrices. Section 5 extends these results from diagonal matrices to irreducible block diagonal matrices in the polynomial case. An important part of this section is devoted to showing the close relation between the graph associated to the representation matrix and the polynomial growth of the order of the average complexity of its operators. As immediate application, we shall detail formal differentiation analyzing the average complexities of derivative operators of order k. The purpose of section 6 is to treat applications using the techniques of this paper. They are based on two recurrent principles: finding the dominant singularity of operator cost series and applying the polynomial growth rule to compute average complexities orders.

2. Operator cost generating function

Here we first define regular rewriting systems and the cost generating function of an operator. Then, we give the most important results concerning the representation matrix of regular rewriting systems.

2.1. **Regular rewriting systems.** Given a set F of operator symbols with arity and a set X of variable symbols, we denote by $\mathcal{T}_{F\cup X}$ the set of terms built from $F \cup X$. Let $\mathrm{Var}(t)$ be the set of variables in term t.

A *rewriting rule* R is a production $g \longrightarrow d$ where g and d are terms and $\mathrm{Var}(g) \subseteq \mathrm{Var}(d)$. A *rewriting system* is a set of rewriting rules.

Rewriting rules define *rewriting relations* between terms: given two terms t and u, t rewrites into u (noted $t \longrightarrow u$) according to rule R iff there exists a subterm of t rooted in m and a substitution $\sigma : X \longrightarrow \mathcal{T}_{F\cup X}$ such that the m rooted subterm of t is equal to $g\sigma$ and u results from the substitution of m for $d\sigma$ in t:

$$t/m = g\sigma \qquad \text{and} \qquad u = t[m \longleftarrow d\sigma]$$

A term is in *normal form* if it cannot be rewritten anymore. Let Ω be a set of constructors and $\mathcal{T}_\Omega$ the family of terms built from Ω.

A *regular* rewriting system is defined as follows:

- every term of $\mathcal{T}_\Omega$ is in normal form.

- the computation of an operator over $\mathcal{T}_\Omega$ arguments always ends with a finite number of rewriting steps on a term in normal form.
- the normal form is unique and the number of rewriting steps does not depend on the strategy.

2.2. **Operator cost.** The *cost of an operator* f on a term $t \in \mathcal{T}_\Omega$ is the number of rewriting steps needed to get its normal form.

Assume that operator f is of arity k. Let C_n^f denote the cumulative cost of f over k-forests of total size n:

$$C_n^f = \sum_{\substack{t_1,\dots,t_k \in \mathcal{T}_\Omega \\ |t_1|+\cdots+|t_k|=n}} cost(f(t_1,\dots,t_k)).$$

If all the k-tuples of size n have the same probability, the *average cost* of f over terms of size n is:

$$\bar{C}_n^f = \frac{C_n^f}{T_n}$$

where T_n is the number of k-tuples $(t_1,\dots,t_k)$ such that for all $1 \leqslant i \leqslant k$, $t_i \in \mathcal{T}_\Omega$ and $|t_1|+\cdots+|t_k| = n$.

We get the evaluation of C_n^f and T_n extracting coefficients from the cost series of f: $C_f(z) = \sum C_n^f z^n$ and from the enumerative series $T(z)$ of $\mathcal{T}_\Omega$.

2.3. **Structure of the rewriting system rules.** Term construction is formalized through Ω-*enumeration* while rewriting rules in an Ω-*definition* are submitted to particular constraints in order to form regular systems.

Definition 2.1. [2] Let E be a set of indices.

A Ω-enumeration is a finite family of n-tuples $(\vec{\omega}_e)_{e\in E}$ of $\mathcal{T}_\Omega^k$ such that:

- each $\vec{\omega}_e$ contains at least one constructor symbol.
- for any $\vec{t} \in \mathcal{T}_\Omega^k$ there exists a unique substitution $\sigma : X \longrightarrow \mathcal{T}_\Omega$ and a unique e such that $\vec{t} = \vec{\omega}_e\sigma$.

A Ω-definition of an operator f is a finite set of rewriting rules $\mathcal{R}_f = (R_e^f)_{e\in E_f}$ such that:

$$R_e^f : f(\omega_e) \longrightarrow D_e$$

where $(\vec{\omega}_e)_{e\in E_f}$ is a Ω-enumeration of $\mathcal{T}_\Omega^k$; and each right member D_e is of the form:

$$D_e = K(x_1,\dots,x_n,\phi_1,\dots,\phi_m).$$

The header K only contains constructor symbols; the x_i's are such that:

$$\{x_1,\dots,x_n\} \subseteq X_e = Var(\vec{\omega}_e)$$

and each ϕ_j is of the form:

$$\phi_j = g(y_1,\dots,y_{r_g})$$

where g is an operator with arity k. Moreover, $\{y_1,\dots,y_{r_g}\} \subseteq X_e$ and $\forall i \neq j,\ y_i \neq y_j$

2.4. **Matrix representation of rewriting systems.** Let us start with some notations: given a regular system rule

$$R_e^i : f_i(\vec{\omega}_e) \longrightarrow D_e$$

we denote by:

- $\epsilon_j(R_e^i)$ the number of times f_j occurs in the right member D_e.
- $b(R_e^i)$ the number of constructor occurrences in $\vec{\omega}_e$.
- $X(R_e^i)$ the number of variable occurrences in $\vec{\omega}_e$.
- r_j the arity of f_j.

Moreover,

$$\vec{C}(z) = (C_{f_1}(z),\dots,C_{f_p}(z))$$

and

$$\vec{Y}(z) = (T^{r_1}(z),\dots,T^{r_p}(z))$$

Theorem 2.1. [2] *The cost series of a regular system can be written:*

$$\vec{C}(z) = M(z).\vec{C}(z) + \vec{Y}(z).$$

where $\vec{C}(z)$ is the cost vector and $\vec{Y}(z)$ is the term enumerating vector. The square matrix $M(z)$ gives the form of the system rules: given $M(z) = (M_{i,j}(z))_{1 \leqslant i,j \leqslant p}$, we have:

$$M_{i,j}(z) = \sum_{R_e^i \in \mathcal{R}_{f_i}} \epsilon_j(R_e^i).z^{b(R_e^i)}.T(z)^{X(R_e^i)-r_j} \tag{1}$$

This result can also be written:

$$\begin{pmatrix} C_{f_1}(z) \\ \vdots \\ C_{f_p}(z) \end{pmatrix} = \begin{pmatrix} M_{1,1}(z) & \cdots & M_{1,p}(z) \\ \vdots & \ddots & \vdots \\ M_{p,1}(z) & \cdots & M_{p,p}(z) \end{pmatrix} \begin{pmatrix} C_{f_1}(z) \\ \vdots \\ C_{f_p}(z) \end{pmatrix} + \begin{pmatrix} T^{r_1}(z) \\ \vdots \\ T^{r_p}(z) \end{pmatrix} \tag{2}$$

Hence, we have:

$$C_{f_i}(z) = \frac{\det((\mathrm{Id} - \mathrm{M})^{[i]}(z))}{\det(\mathrm{Id} - \mathrm{M}(z))} \tag{3}$$

where **Id** is the identity matrix and $\mathbf{(Id\text{-}M)^{[i]}(z)}$ is the $\mathbf{(Id\text{-}M)(z)}$ matrix where the i^{th} column is replaced by $\vec{Y}(z)$.

Example 3. Let us take our first example with operators *sin* and *cos*. Its representation matrix is:

$$M = \begin{array}{c} \scriptstyle\sin \\ \\ \scriptstyle\cos \end{array} \begin{pmatrix} 2zB(z) & 2zB(z) \\ & \\ 2zB(z) & 2zB(z) \end{pmatrix}$$

where $B(z)$ stands for the binary trees enumerative series. Hence, we have:

$$C_{\sin}(z) = C_{\cos}(z) = \frac{B(z)(1 + B^2(z))}{3B^2(z) - 1} \ \blacklozenge$$

Example 4. Our simple differentiation system of example 2 yields this representation matrix with the same enumerative series $B(z)$:

$$M = \begin{array}{c} \scriptstyle d \\ \\ \scriptstyle cp \end{array} \begin{pmatrix} 2zB(z) & 2zB(z) \\ & \\ & 2zB(z) \end{pmatrix}$$

Applying the preceding theorem gives:

$$C_{cp}(z) = \frac{B(z)}{1 - 2zB(z)} \qquad \text{and} \qquad C_d(z) = \frac{B(z)}{(1 - 2zB(z))^2} \ \blacklozenge$$

Example 5. [2] This example shows a shuffle of trees with two operators f and g. Terms are binary trees constructed from binary symbol o and constant a.

Let $B(z)$ denote the enumerating series of binary trees.

$$\begin{array}{cccc}
R_1^f & f(a,a) & \longrightarrow & a \\
R_2^f & f(o(u,v),a) & \longrightarrow & g(u,v) \\
R_3^f & f(a,o(u,v)) & \longrightarrow & g(u,v) \\
R_4^f & f(o(u_1,v_1),o(u_2,v_2)) & \longrightarrow & o(f(u_1,u_2),f(v_1,v_2)) \\
\\
R_1^g & g(a,a) & \longrightarrow & a \\
R_2^g & g(a,o(u,v)) & \longrightarrow & g(u,v) \\
R_3^g & g(o(u,v),a) & \longrightarrow & g(u,v) \\
R_4^g & g(o(u_1,v_1),o(u_2,v_2)) & \longrightarrow & o(f(u_1,u_2),g(v_1,v_2))
\end{array}$$

The representation matrix $M(z, B(z))$ is:

$$M = \begin{array}{c} f \\ \\ g \end{array} \begin{pmatrix} 2z^2B^2(z) & 2z^2 \\ & \\ z^2B^2(z) & 2z^2 + z^2B^2(z) \end{pmatrix}$$

We easily deduce both $C_f(z)$ and $C_g(z)$ with formula 3:

$$C_f(z) = C_g(z) = \frac{B^2(z)}{1 - 2z^2 - 2z^2B^2(z)} \blacklozenge$$

3. Average complexity

We shall first give an overview of some results concerning the enumeration of trees in order to state two key theorems about the asymptotic behavior of the average complexity of the operators.

3.1. **Counting trees.** This section briefly recalls the singularity analysis methods to estimate the coefficients of the enumeration series of simple families of trees. Let $T(z)$ denote this series.

$$T(z) = \sum_{t \in \mathcal{T}} z^{|t|} = \sum_n T_n z^n$$

where T_n is the number of trees of size[1] n. As usual, $[z^n]T(z)$ denotes the coefficient of z^n in $T(z)$.

Theorem 3.1. [5, 8] *Let $T(z)$ be a series defined as follows:*

$$T(z) = z\Phi(T(z))$$

where Φ is an aperiodic[2] polynomial function such that $\Phi(0) \neq 0$ and $\Phi'(0) = 0$.

Then, $T(z)$ admits the following expansion around $z = \rho$:

$$T(z) = h(z) + g(z)\sqrt{1 - \frac{z}{\rho}}$$

where h and g are analytic and $h(\rho) = \tau$ and $g(\rho) = -\sqrt{\frac{2\Phi(\tau)}{\Phi''(\tau)}}$

Moreover,

$$T_n = \sqrt{\frac{\Phi(\tau)}{2\pi\Phi''(\tau)}}\rho^{-n} n^{-3/2}\left(1 + O\left(\frac{1}{n}\right)\right) \tag{4}$$

Sketch of proof. Let ρ be the dominant singularity of $T(z)$ and τ be the limit of $T(z)$ at ρ.

ρ is the point of smallest module where $T(z)$ ceases to be analytic. Indeed, ρ is defined by the system:

$$\begin{cases} \tau\Phi'(\tau) - \Phi(\tau) = 0 & \tau \text{ being the smallest real positive root} \\ \\ \rho = \dfrac{\tau}{\Phi(\tau)} \end{cases} \tag{5}$$

It follows that $\rho < 1$.

To get the expansion of $T(z)$ around $z = \rho$, we expand $u/\Phi(u)$ around τ. That gives:

$$z = \frac{\tau}{\Phi(\tau)} + \sum_{n \geqslant 2} c_n(z - \tau)^n \qquad \text{with} \qquad c_2 = -\frac{\tau\Phi''(\tau)}{2\Phi'(\tau)}$$

Then, we deduce the expansion of $T(z)$ around $z = \rho$:

$$T(z) = \tau + \sum_{n \geqslant 1} e_n \left(1 - \frac{z}{\rho}\right)^{n/2}$$

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[1] When applied to term trees, it denotes its number of nodes

[2] i.e. there is no polynomial Ψ and no integer $d > 1$ such that $\forall y\ \Phi(y) = \Psi(y^d)$.

thus:

$$T(z) = h(z) + g(z)\sqrt{1-\frac{z}{\rho}}$$

with g and h analytic.

Applying singularity analysis leads to formula (4). □

Example 6. We treat now the example of unary-binary trees, where $B(z)$ denotes their enumerative series.

A tree is defined recursively: from a node, we can attach a terminal node (a *leaf*); one tree if the node is of arity 1; two trees if the node is of arity 2.

Since the size of a node is 1, we have:

$$B(z) = z + zB(z) + zB^2(z) = z(1 + B(z) + B^2(z))$$

Further computations give:

$$B(z) = \frac{1-z-\sqrt{1-2z-3z^2}}{2z} = \frac{1-z}{2z} - \frac{\sqrt{1+z}}{2z}\sqrt{1-\frac{z}{3}}$$

The dominant singularity is $z = 1/3$. ♦

Example 7. Take the unary-binary trees enumerated by $B(z)$. We solve the system 5 and use relation 4. We find:

$$\tau = 1 \text{ and } \rho = 1/3$$

therefore:

$$[z^n]B(z) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\left(\frac{1}{3}\right)^{-n} n^{-3/2}\left(1 + O\left(\frac{1}{n}\right)\right) \text{ ♦}$$

Remark 1. If $T(z)$ have multiple singularities on its circle of convergence[3] (take binary trees as an example), contributions add themselves and this does not change the asymptotic behavior of $[z^n]T(z)$. See works of Flajolet and Sedgewick [4] for details.

Example 8. Let $B(z) = z(1 + B^2(z))$ be the enumerative series of binary trees. Its singularities are gathered in the equation: $\tau^2 = 1$. Consequently, we have:

$$\begin{array}{ll} B_n = \frac{1}{\sqrt{2\pi}} 2^{n+1} n^{-3/2}\left(1 + O\left(\frac{1}{n}\right)\right) & n \text{ even} \\ B_n = 0 & n \text{ odd} \end{array}$$

3.2. **Average cost of operators.** Since we want to estimate the average cost $\bar{C}_n^{f_i}$ of operator f_i, we have to evaluate $[z^n]C_{f_i}(z)$ as $n \to \infty$. This work has been done in [2] and it leads to the following results.

First notice that for any enumerative series of simple families of trees $T(z)$, we have: $z = \frac{T(z)}{\Phi(T(z))}$. Since Ω is a finite set of constructor symbols with finite arities, Φ is always a polynomial function. Thus, equation 3, giving the cost function of an operator f_i, is of the form:

$$C_{f_i}(z) = \frac{P_i(T(z))}{Q_i(T(z))}$$

where $T(z)$ is the enumerative series of $\mathcal{T}_\Omega$ and P_i, Q_i are prime polynomials with integer coefficients.

Here is the case separating theorem stating the supposed average complexities behaviors according to the position of the singularity of $T(z)$ with respect to the zeros of $Q_i(T(z))$. .

Theorem 3.2. [2] *Let f_i be a regular system operator with cost generating function:*

$$C_{f_i}(z) = \frac{P_i(T(z))}{Q_i(T(z))}$$

where P_i and Q_i are prime polynomials. Let ρ be $T(z)$ dominating singularity and let ρ_{0_i} be the smallest positive root of $Q_i(T(z))$.

[3] This occurs precisely when Φ is aperiodic

- *if* $\rho < \rho_{0i}$ *then* $\bar{C}_n^{f_i} = c_1(1 + O(1/n))$
- *if* $\rho = \rho_{0i}$ *then* $\bar{C}_n^{f_i} = c_2 n^{q/2}(1 + O(1/\sqrt{n}))$
- *if* $\rho > \rho_{0i}$ *then* $\bar{C}_n^{f_i} = c_3 \alpha^n n^{r+1/2}(1 + O(1/n))$

where the c_i*'s are real constants that depend on* P_i*,* Q_i *and* T*, and* q *and* r *are positive integers.*

Remark 2. With singularity analysis, we can give the order of growth of such operators in terms of the multiplicity of factor $(B(z) - \tau)$ in the denominator of operator cost series. That is, with the same notations as in theorem 3.2, assume there exists operator f_i such that $C_{f_i}(z)$ is of the form :

$$\frac{P_i(B(z))}{Q_i(B(z))} = \frac{1}{(B(z) - \tau)^s} \frac{P_i(B(z))}{\bar{Q}_i(B(z))}$$

where s is a strictly positive integer and $\bar{Q}_i$ a polynomial such that $\bar{Q}_i(\tau) \neq 0$. Expanding the preceding equality around τ we see that its leading term is in $(1 - z/\rho)^{-s/2}$. Thus, applying the transfer lemma:

$$C_n^{f_i} = \frac{(-1)^s}{\Gamma(s/2)} \frac{P_i(\tau)}{\bar{Q}_i(\tau)} \left(\frac{2\Phi(\tau)}{\Phi''(\tau)}\right)^{-s/2} \rho^{-n} n^{s/2-1} \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)$$

Dividing this expression by the asymptotic evaluation of $[z^n]T^{r_i}(z)$, we find that the order of growth of $\bar{C}_n^{f_i}$ is $O(n^{\frac{s+1}{2}})$.

Example 9. Let us take the shuffle defined in the preceding example. We found

$$C_f(z) = C_g(z) = \frac{B^2(z)}{1 - 2z^2 - 2z^2 B^2(z)}$$

Replace z by $B(z)/\Phi(B(z))$:

$$C_f(z) = C_g(z) = \frac{B^2(z)(1 + B^2(z))}{(1 - B(z))(1 + B(z))}$$

Note that $\tau_0 = 1 = \tau$ therefore $\rho_0 = \rho$ so we are in the polynomial case.

From the previous remark we deduce that $\bar{C}_n^f = \bar{C}_n^g = O(n)$. ♦

4. Structure of representation matrices

Studying the representation matrix structure will precise the results of theorem 3.2. It consists in determining the dominant singularity of any operator cost series isolating each group of operator according to their mutual dependencies. This notion is translated graphically into strongly connected components in which each operator cost series has the same radius of convergence.

4.1. **Strategy.** Let $M(z, T(z))$ be the representation matrix of some regular rewriting system and

$$C_{f_i}(z) = \frac{\det((\mathrm{Id} - \mathrm{M})^{[i]}(z))}{\det(\mathrm{Id} - \mathrm{M}(z))}$$

the cost series associated to operator f_i.

To evaluate $[z^n]C_{f_i}(z)$, we have to look for the dominant singularity of $C_{f_i}(z)$, that is, the smallest positive root of $\det(\mathrm{Id}-M)$.

The idea is to turn M to a triangular block matrix – whenever it is possible – through a suitable permutation matrix.

$$\begin{pmatrix} A_{1,1} & & & 0 \\ A_{2,1} & A_{2,2} & & \\ \vdots & \ddots & \ddots & \\ A_{k,1} & \ldots & A_{k,k-1} & A_{k,k} \end{pmatrix} \tag{6}$$

Then, we study the dominating eigenvalue of each block – remember that $\det(\mathrm{Id}-M) = \Delta(1)$ where Δ is the characteristic polynomial of M. So, we have to prove that there exists a positive dominating eigenvalue in each block $A_{i,i}$ corresponding to the smallest positive root of $\det(\mathrm{Id}-A_{i,i})$.

Since $\det(\mathrm{Id}\text{-}M) = \prod_{i=1}^{k} \det(\mathrm{Id}-A_{i,i})$, we will be able to deduce the order of the asymptotic expansion of $\bar{C}_n^{f_i}$ for each operator f_i.

When matrix M is of the form above, we say that M is *reducible*.

Definition 4.1. Let B be a square matrix. B is reducible iff there exists a permutation matrix P such that:

$$PBP^T = \begin{pmatrix} B_1 & 0 \\ B_2 & B_3 \end{pmatrix}$$

where B_1 and B_3 are square matrices.

Otherwise B is said to be *irreducible*.

Recursively applying this process to each block, we obtain matrices of the form (6) where each diagonal block $A_{i,i}$ is irreducible.

With this notion and the nonnegativity property of representation matrices, we can use the Perron-Frobenius theory to prove the existence of positive dominating eigenvalues in each diagonal block.

The results below are valid for irreducible matrices, as if we were focusing on a diagonal square block $A_{i,i}$.

4.2. Irreducible matrices.

Lemma 1. (Frobeniüs [1, 7]) *An irreducible nonnegative matrix A has a real positive eigenvalue r such that:*

$$r > |\lambda_i|$$

for any eigenvalue $\lambda_i \neq r$ of A. Furthermore, there is a positive eigenvector corresponding to r.

Theorem 4.1. *Let $M(z, T(z))$ denote a representation matrix. If M is irreducible then there exists a unique function $r(z, T(z))$ with values in $\mathbb{R}$ such that:*

- $\forall z \in [0, \rho]$ $r(z, T(z))$ *is an eigenvalue of* $M(z, T(z))$
- $\forall z \in [0, \rho]$, $\forall k$, $r(z, T(z)) > |\lambda_k(z, T(z))|$

where the λ_i's are all the eigenvalues of M different from r.

Proof. M is a representation matrix therefore all the $M_{i,j}$'s are polynomials in $z, T(z)$ with positive integer coefficients. Since $T(z)$ is positive for $z \in [0, \rho]$, $M(z, T(z))$ is a nonnegative irreducible matrix. Consequently, from lemma 1 there exists a dominating positive eigenvalue $r(z, T(z))$ for all $z \in [0, \rho]$.

This eigenvalue is unique for all $z \in [0, \rho]$: assume there exists $I \subset [0, \rho]$ such that for all $z \in I$ there exists another dominating eigenvalue $r_2(z, T(z))$. This means that for some z_0, we have $r(z_0) = r_2(z_0)$, and contradicts with the irreducibility hypothesis of M. $\square$

The next theorem gives an easier way to spot the smallest root of $\det(\mathrm{Id}-M)$ on the real positive axis: instead of computing $r(z, T(z))$, we just look for the highest eigenvalue of matrix $M(\rho, \tau)$.

Definition 4.2. If $C \in \mathcal{M}_n(\mathbb{C})$ and $A = (a_{i,j})$ is a nonnegative matrix of $\mathcal{M}_n(\mathbb{C})$ such that $|C| \leqslant A$ (i.e. $|c_{i,j}| \leqslant a_{i,j}$ for all i, j), then A is said to dominate C.

Lemma 2. (Wielandt [1, 7]) *If a complex matrix C is dominated by an irreducible matrix A with maximal eigenvalue r, then for every eigenvalue s of C:*

$$|s| \leqslant r$$

Lemma 3. *Let $M(z, T(z))$ be an irreducible representation matrix and $r(z, T(z))$ denote the dominating eigenvalue of M.*

Then $r(z, T(z))$ grows strictly as $z \in [0, \rho]$ and $\lim_{z \to +\infty} r(z, T(z)) = \infty$.

Proof. Let z_0 and z_1 be two points of $[0, \rho]$ such that $z_0 < z_1$. Since $T(z)$ grows as $z \in [0, \rho]$, the $M'_{i,j}s$ are increasing functions of z. So, we have:

$$0 < M(z_0, T(z_0)) < M(z_1, T(z_1)).$$

Therefore: $r(z_0, T(z_0)) < r(z_1, T(z_1))$.

The theory of nonnegative matrices gives us a lower bound for $r(z, T(z))$; namely we have for instance:

$$0 < \min_j \sum_i M_{ij}(z, T(z)) \leqslant r(z, T(z)).$$

Since the entries of M are polynomials in z and $T(z)$ with positive coefficients, so is the quantity $\min_j \sum_i M_{ij}(z, T(z))$. Hence the result. □

Lemma 4. *Let $M(z, T(z))$ be an irreducible representation matrix. Let ρ_0 denote the smallest positive root of det(Id−M) and ρ_{0i} denote the dominant singularity of $C_{f_i}(z)$.*

Then, for each operator f_i, $\rho_0 = \rho_{0i}$.

Proof. The easiest way seems to show that $\tau_0 = \tau_{0i}$. It follows that $\rho_0 = \rho_{0i}$ since $u/\Phi(u)$ is a strictly increasing function on the positive real axis.

It is necessary to show that $(T(z)-\tau)$ does not divide $\det(\mathrm{Id}-M)$. Thus, since the numerator of $C_{f_i}(z)$ is a polynomial which turns to be in $T(z)$, we just have to prove that $\det(\mathrm{Id}-M)^{[i]}(\rho_0, \tau_0) \neq 0$.

$$\det(\mathrm{Id}-M)^{[i]}(z, T(z)) = \begin{pmatrix} 1-M_{1,1} & -M_{1,2} & \cdots & Y_1 & \cdots & -M_{1,n} \\ -M_{2,1} & 1-M_{2,2} & \cdots & Y_2 & \cdots & -M_{2,n} \\ \vdots & \vdots & & \vdots & & \vdots \\ -M_{n,1} & -M_{n,2} & \cdots & Y_n & \cdots & 1-M_{n,n} \end{pmatrix}$$

Expanding $\det(\mathrm{Id}-M)^{[i]}(z, T(z))$ by the ith column, we get:

$$\det(\mathrm{Id}-M)^{[i]}(\rho_0, \tau_0) = \sum_{j=1}^{n} Y_i B_{i,j}(1) \tag{7}$$

where $B(x)$ is the adjoint matrix of $(x\mathrm{Id}-M)$: $\mathrm{adj}(x\mathrm{Id}-M)$. The dominating eigenvalue of $M(\rho_0, \tau_0)$ being $r(\rho_0, \tau_0)$, $B(r(\rho_0, \tau_0))$ is a positive matrix; thus all its entries are strictly positive. Since $\vec{Y}(z)$ is a positive vector, we have, from relation (7):

$$\det(\mathrm{Id}-M)^{[i]}(\rho_0, \tau_0) \neq 0$$

So, $(T(z)-\tau)$ does not divide $\det(\mathrm{Id}-M)^{[i]}(z, T(z))$.

Finally, $\rho_0 = \rho_{0i}$. □

Corollary 1. *Let $M(z, T(z))$ denote an irreducible representation matrix.*

Then the average complexity of every operator f_i of M has the same order of growth.

Proof. Since for all i, $C_{f_i}(z)$ has a unique dominant singularity, we just apply the results of singularity analysis. □

Theorem 4.2. *Let $M(z, T(z))$ be an irreducible representation matrix of some regular rewriting system. Let ρ_0 denote the smallest positive root of det(Id−M) and ρ be the dominant singularity of $T(z)$. Let $r(z, T(z))$ be the dominating eigenvalue of M.*

- *$r(\rho, \tau) = 1$ iff $\rho_0 = \rho$.*
- *$r(\rho, \tau) > 1$ iff $\rho_0 < \rho$.*

Proof. Since

$$\det(\mathrm{Id}-M) = \prod_k (1-\lambda_k)^{\alpha_k}$$

where the λ_k's are the eigenvalues of M and the α_k's are their respective multiplicities, finding the smallest root of $\det(\mathrm{Id}-M)$ consists in comparing $r(z, T(z))$ with 1.

First, assume $r(\rho, \tau) = 1$; ρ is a root of $\det(\mathrm{Id}-M)$. By lemma 3, r grows strictly on $[0, \rho]$. Therefore, there is no other $\rho_1 < \rho$ such that $r(\rho_1, \tau_1) = 1$. Furthermore r is the dominating eigenvalue, so we have $\rho_0 = \rho$. Conversely, if $\rho_0 = \rho$, $\det(\mathrm{Id}-M)(\rho,\tau) = 0$; thus for some k_0, we have $\lambda_{k_0}(\rho,\tau) = 1$. As $\rho_0 = \rho$, we have $\lambda_{k_0} = r$ for r is unique and grows strictly over $[0,\rho]$.

Next, assume $r(\rho,\tau) > 1$. Since r strictly grows on $[0,\rho]$, this implies that $\rho_0 < \rho$. Reciprocally, if $\rho_0 < \rho$, for the same reasons, we have: $r(\rho_0,\tau_0) = 1$. Moreover, r grows strictly therefore $r(\rho,\tau) > 1$. □

Example 10. Take example 1 with operators *sin* and *cos*. We recall that $C_{\cos}(z) = C_{\sin}(z) = B(z)/(1-4zB(z))$ where $B(z)$ stands for the binary trees enumerative series. The dominating eigenvalue function $r(z,B(z))$ of its representation matrix is:

$$r(z,B(z)) = 4zB(z)$$

We notice as expected that $\det(\mathrm{Id}-M) = 1 - r(z,B(z))$ corresponds to both $C_{\sin}(z)$ and $C_{\cos}(z)$ denominators.

Since $r(\rho,\tau) = 2$, we are sure that there exists some ρ_0 such that $0 < \rho_0 < \rho$. We have there a simple trick to find out that the average complexities of both operators have an exponential order of growth. ♦

5. Polynomial case

The polynomial case corresponds to a critical point where the dominant singularity of the cost series is equal to the singularity of the enumerative series of the $\mathcal{T}_\Omega$ terms, namely: $\rho_0 = \rho$.

We shall see in the sequel that the order of growth of the average complexity of any operator f_i can be determined quite easily using the methods of the preceding section. The first part deals with irreducible representation matrices and then the reducible case will be discussed using the example of formal derivation.

5.1. **Order of operators in an irreducible matrix.**

We show that the average case complexity of any operator in a rewriting system represented by an irreducible matrix is generally linear in the size of the term trees, with a pathological case in $n^{3/2}$.

Lemma 5. (Frobeniüs [7]) *The maximal eigenvalue of an irreducible nonnegative matrix is a simple root of its characteristic equation.*

Theorem 5.1. *Let $M = (M_{i,j}(z,T(z)))$ be an irreducible representation matrix and $r(z,T(z))$ be its dominating eigenvalue. Assume $r(\rho,\tau) = 1$*

- $\bar{C}_n^{f_k} \sim c_{f_k} n^{3/2}$ *if no entry of M depends on $T(z)$.*
- $\bar{C}_n^{f_k} \sim c_{f_k} n$ *if there exists at least one entry of M that depends on $T(z)$.*

where the c_{f_k}'s are constants depending on f_k.

Proof. In the first case of the theorem, we assume that the representation matrix M has entries in z only. Therefore, there exists an algebraic function λ that is solution of the characteristic polynomial $\chi(z,\mu) = \det(\mu\mathrm{Id} - M(z))$ and such that $\lambda = r$ for $z > 0$. As we have seen before, the dominant singularity of $\det(\mathrm{Id} - M(z))$ in the polynomial case is $z = \rho$, therefore, since $r(\rho) = 1$, we expand $\det(\mathrm{Id} - M(z))$ around ρ and use the transfer lemma to get the asymptotic estimation of $\bar{C}_n^{f_k}$.

The roots of $\det(\mathrm{Id} - M(z))$ are the poles of $C_{f_k}(z)$. These roots correspond to the solutions of the equation $r(z) = 1$, with $z \in (0,\rho]$. Indeed, we have:

$$\det(\mathrm{Id} - M(z)) = (1 - r(z))Q(z)$$

where

$$Q(z) = \prod_{k=1}^{\dim M(z)-1} (1 - \lambda_k(z))$$

with the λ_k's being the eigenvalue functions of $M(z)$. From the Perron-Frobeniüs theorem, we know that

$$\forall k, |\lambda_k(\rho)| < r(\rho) = 1.$$

Consequently, $Q(\rho) \neq 0$.

Since $\lambda(z)$ is an algebraic function around $z=\rho$, we know from the implicit function theorem that $\lambda(z)$ is analytic over a neighborhood of ρ. But before applying the transfer lemma, we have to determine the order of ρ as solution of the equation $r(z)=1$. So, we need to study the successive derivatives of λ at ρ. In that case, we prove that $\lambda'(\rho)\neq 0$, that is, ρ is a solution of order 1 of the equation $r(z)=1$. Though this result has already been proved in the general case as a corollary of the implicit function theorem (see for instance [6]), our proof uses Perron-Frobenius theory to establish this property.

We shall assume $\lambda'(\rho)=0$ and show a contradiction. We recall that $r(z)$ grows for $z>0$. We make here a few remarks:

- if $\lambda'(\rho)=0$ then the first non zero derivative at ρ has odd order. Indeed, the local behavior of λ would contradict the fact that r grows strictly.
- Thus there exists integer $k>0$ such that the expansion of λ around $z=\rho$ yields
$$\lambda(z)=1+\sum_{j\geqslant 2k+1} c_j(z-\rho)^j,$$
where $c_j>0$ for all j. For convenience, we assume $c_3\neq 0$, but the end of this proof would still hold with $\lambda^{(k)}(\rho)=0$ for all integer $0<k<N$, where N is an odd integer.

Locally, $|\lambda(\rho+xe^{i\theta})|$ is a growing function of $x>0$ along the branches $\theta_1=2i\pi/3$ and $\theta_2=-2i\pi/3$.

We consider $\zeta=\rho+xe^{\frac{2i\pi}{3}}$ such that $|\zeta|<\rho$. We have:
$$|\lambda(\zeta)|>\lambda(\rho)=1$$

Moreover, $M(|\zeta|)$ dominates $M(\zeta)$. Indeed, from the triangular inequality: $\forall i,j\ |M_{i,j}(\zeta)|<M_{i,j}(|\zeta|)$ since all the $M_{i,j}$'s are positive coefficiented polynomials. We apply now Wielandt's theorem (see lemma 2):

Since $\lambda(\zeta)$ is an eigenvalue of $M(\zeta)$ and $M(|\zeta|)$ dominates $M(\zeta)$ then we have:
$$\lambda(|\zeta|)>|\lambda(\zeta)|>\lambda(\rho)=1,$$
that is:
$$r(|\zeta|)>r(\rho)=1.$$
This contradicts:
$$|\zeta|<\rho \text{ and } r \text{ grows over } (0,\rho].$$

Consequently, around $z=\rho$ the expansion of $\det(\mathrm{Id}-M(z))$ is:
$$\det(\mathrm{Id}-M(z))=(\rho-z)Q(z)[r'(\rho)+\sum_{k>0}c_k(\rho-z)^k]$$
that is:
$$\det(\mathrm{Id}-M(z))=(1-z/\rho)Q(z)[\rho r'(\rho)+\sum_{k>0}e_k(1-z/\rho)^k]$$
As previously seen, the numerator of $C_{f_k}(z)$ does not vanish at $z=\rho$, therefore, applying singularity analysis we have:
$$C_n^{f_k}\sim\gamma_k\rho^{-n}.$$
Then, dividing by T_n gives:
$$\bar{C}_n^{f_k}\sim c_{f_k}n^{3/2}.$$

Now, we study the second case of the theorem. Let $P(z,y)=\det(\mathrm{Id}-M(z,y))$. Since matrix $M(1,y)$ has the same structure as matrix $M(z,1)$ (i.e. the univariate matrix in z), for the same reasons as above, we have $P'_y(\rho,\tau)\neq 0$. Substituting y by $T(z)$ around $z=\rho$ gives:
$$P(z,T(z))=\alpha(1-z/\rho)^{1/2}+O((1-z/\rho))$$
with $\alpha\neq 0$. Applying singularity analysis, we have:
$$C_n^{f_k}\sim\beta_k\rho^{-n}n^{-1/2}.$$
Then, dividing by T_n gives:
$$\bar{C}_n^{f_k}\sim c_{f_k}n.$$

$\square$

The reader may report to examples 9 to see what happens in the non pathological case before examining the following example:

Example 11. Consider matrix:

$$M(z,T(z)) = \begin{pmatrix} 2z & z \\ z & 2z \end{pmatrix}$$

where term trees are unary-binary trees enumerated by $T(z) = z(1 + T(z) + T^2(z))$. We recall that:

$$\tau = 1 \quad \text{and} \quad \rho = 1/3$$

Let us compute $\det(\mathrm{Id}-M)$:

$$\det(\mathrm{Id} - M) = (1 - 3z)(1 - z) = \frac{(T(z) - 1)^2}{\Phi^2(T(z))}$$

It appears that the multiplicity of factor $(T(z)-\tau)$ is 2. Consequently all the operators represented by M will have an average complexity of order $n^{3/2}$. ⧫

5.2. **Graphical translation of representation matrices.** This section is an introduction to the study of systems represented by reducible matrices. Their associated graph is of precious help since we will be able to draw out a hierarchy between operators.

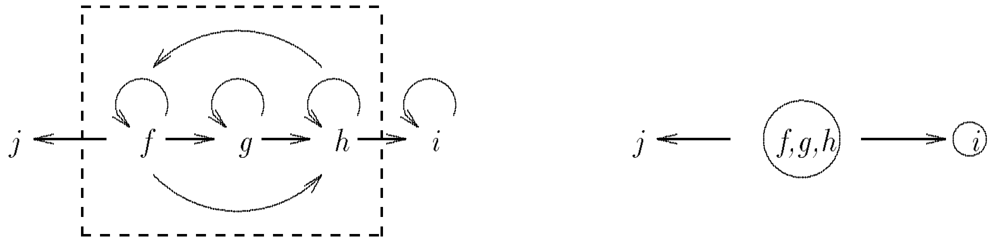
Let $\mathcal{G}_M$ be the directed graph of matrix $(M_{i,j})$. This graph is built as follows: put an arrow from vertex j to vertex i if $M_{i,j} \neq 0$.

Theorem 5.2. *M is an irreducible matrix if and only if $\mathcal{G}_M$ is strongly connected.*

This theorem brings out the interest of graphs in the computation of operators average complexities. Since we have shown that the average complexity of all the operators represented by irreducible matrices have the same order of growth, we will just have to symbolize such blocks by a single vertex.

From now on, the ★ symbol will denote a non identically 0 block that is irreducible on the diagonal.

Example 12. Assume matrix M is of the form: $\begin{array}{c} f \\ g \\ h \\ i \\ j \end{array} \begin{pmatrix} \star & 0 & \star & 0 & 0 \\ \star & \star & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 \\ 0 & 0 & \star & \star & 0 \\ \star & 0 & 0 & 0 & 0 \end{pmatrix}$. Its associated graph is:



strongly connected component

simplified graph

We can see the functions that have the same order of growth (here f, g and h have the same order of growth). We define there two kinds of vertices. The first one is a "round vertex" where the operators that form a strongly connected component are put together in a circle. In the case of a single operator block, this representation is valid when this operator depends on itself; otherwise there is no circle around it. ⧫

5.3. **Reducible matrices.** We recall that matrix M is reducible if, through a suitable permutation matrix P, M is turned into a matrix of the form:

$$\begin{pmatrix} A_{11} & & & \\ A_{21} & A_{22} & & 0 \\ \vdots & & \ddots & \\ A_{n1} & \dots\dots\dots\dots & & A_{nn} \end{pmatrix}$$

the $A_{i,i}$'s being square blocks.

Note that simplified graphs are DAGs with operators (or set of operators) as vertices. The idea is to find inductively the order of growth of the system operators from the roots of the graph to the operators block we need to draw the common cost series dominant singularity. Let us start by the most elementary block-triangular matrices.

5.3.1. *Triangular matrices.* Here diagonal blocks are actually single entries, the simplest case of triangular block matrices.

We compute by induction on $\dim(M)$ the operators average complexities turning the operators cost series into a form that involves the operators cost series of the preceding rank.

Definition 5.1. An operator g is said to be a *predecessor* of operator f if $M_{fg}(z, T(z))$ is not identically zero.

This definition formalizes the condition that there exists a rewriting system rule where f depends on g.

Lemma 6. *Let $\{f_1, \dots, f_n\}$ be a set of system operators such that it can be represented by a triangular matrix M. Let $\{f_{i_1}, \dots, f_{i_k}\}$ with $1 \leqslant i_1 < \dots < i_k \leqslant n$ be the set of all the predecessors of operator f_j, $i_k < j \leqslant n$.*

Then, $C_{f_j}(z)$ admits the following expansion:

$$C_{f_j}(z) = \frac{M_{f_j,f_{i_1}}(z)}{1 - M_{f_j,f_j}(z)} C_{f_{i_1}}(z) + \dots + \frac{M_{f_j,f_{i_k}}(z)}{1 - M_{f_j,f_j}(z)} C_{f_{i_k}}(z) + \frac{T^{r_j}(z)}{1 - M_{f_j,f_j}(z)}$$

Proof. This expression arises from the formula 2 after expanding the line corresponding to operator f_j. □

This expansion infers the following theorem:

Theorem 5.3. *Let τ be the limit value of $T(z)$ at $z = \rho$. With the same hypothesis as the previous lemma, assume for all i_λ, $C_{f_{i_\lambda}}(z) = \frac{P_\lambda(T(z))}{(T(z)-\tau)^{\alpha_\lambda} Q_\lambda(T(z))}$ such that for all λ, P_λ and Q_λ are prime polynomials. Let $\alpha = \max_\lambda\{\alpha_\lambda\}$.*

- *If M_{f_j,f_j} depends on $T(z)$ then $C_{f_j}(z) = \frac{P_j(T(z))}{(T(z)-\tau)^{\alpha+1} Q_j(T(z))}$*
- *If M_{f_j,f_j} does not depend on $T(z)$ then $C_{f_j}(z) = \frac{P_j(T(z))}{(T(z)-\tau)^{\alpha+2} Q_j(T(z))}$*

where the P_j's and Q_j's are prime polynomials and for all j, $Q_j(\tau) \neq 0$.

Proof. We are here interested in the expansion of $(1 - M_{f_j,f_j})$ in terms of $T(z)$. This computation can be lead considering M_{f_j,f_j} as a one-entry irreducible matrix, thus we just apply the results of theorem 5.1 to extract the right power of factor $(T(z) - \tau)$ from the expansion of $(1 - M_{f_j,f_j})$.

Moreover, since all the P_λ's and Q_λ's are prime polynomials, and clearly $(T(z) - \tau)$ does not divide the P_λ's, the $M_{f_j,f_{i_\lambda}}$'s and $T^{r_j}(z)$ we can find two possible expressions for $C_{f_j}(z)$ arising from M_{f_j,f_j} dependency on $T(z)$. □

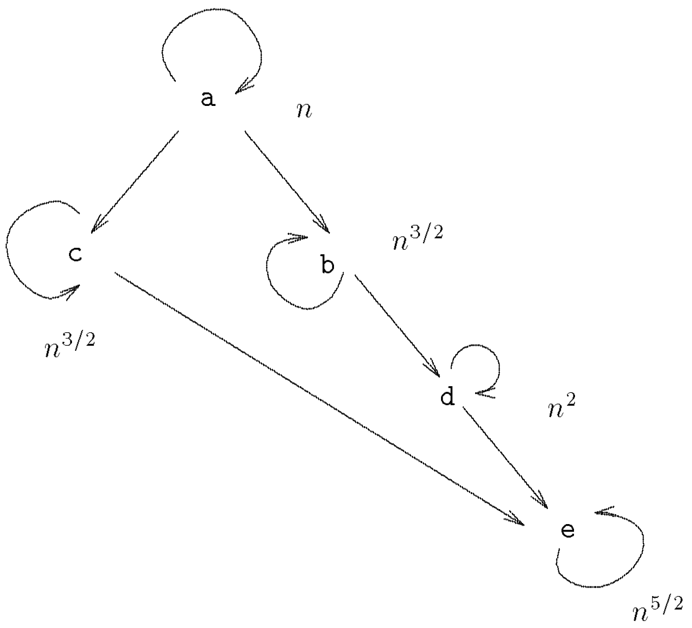
Graphically, the average complexity of any operator f is computed recursively starting from the root of the graph and evaluating the order of the average complexity at each node. In pure polynomial cases, if the path to f is not unique, the path that leads to the right average complexity order in most of the cases (depending on local pathological case) is the longest one.

Example 13. Suppose you have five operators a, b, c, d, e represented by a matrix of the form:

$$\begin{array}{c}a\\b\\c\\d\\e\end{array}\begin{pmatrix}\bigstar &&&&\\ \bigstar & \bigstar &&&\\ \bigstar & 0 & \bigstar &&\\ 0 & \bigstar & 0 & \bigstar &\\ 0 & 0 & \bigstar & \bigstar & \bigstar\end{pmatrix}$$

We assume that we are in the polynomial case and all the diagonal entries of M depend on $T(z)$.

The corresponding graph of M is:



The average complexity of e is of order $n^{5/2}$. It corresponds to the longest path from a to e. ⧫

In order to bring up formal differentiation, we introduce *chain structured systems.*

Definition 5.2. A chain structured system is a regular rewriting system with rewriting operators $f_1, \dots, f_n$ such that:

- For all $1 \leqslant i < n$, f_{i+1} depends on f_i.
- For all i and $j > i$, f_i does not depend on f_j.
- For all i, f_i depends on f_i.

These constrained systems admit a particular representation matrix: if we consider a lower triangular matrix, no entry on the diagonal is zero as well as the entries of the diagonal right below. Consequently from lemma 6 and starting from n at the root, each operator average complexity order of growth is inductively multiplied by n or $\sqrt{n}$ at each vertex moving down the chain.

5.3.2. *Formal differentiation.* This part is devoted to the study of formal differentiation analyzing the average complexity of the kth derivative operator with copy.

Let d_k be the kth derivative operator. Let R_e^k denote the rule $d_k(\omega_e) \longrightarrow D_e$. The following conditions are imposed to formal differentiation systems:

- For each k, the arity of d_k is 1.
- For all $e \in E_{d_k}$, there is only one occurrence of $d_k(x_i)$ in D_e for each $x_i \in X(R_e^k)$.
- There exists an occurrence of $d_{k-1}(x_i)$ for some $x_i \in X(R_e^k)$ for any integer $k > 1$ among the right members of the R_e^k's.
- For all $e \in E_{d_k}$ there is no occurrence of d_i with $i > k$ in the right member D_e.

The two last conditions simply mean that for some $\alpha \in \Omega$, $d_k(\alpha(x_1, \ldots, x_{r(\alpha)}))$ is in terms of $d_{k-1}(x_i)$ and not in terms of any of the $d_j(x_i)$'s for $j > k$. This formalizes what is encountered in the classical mathematical differentiation.

Besides, from the second condition, for each i: $M_{ii}(z) \neq 0$. Thus, matrix $M(z)$ is of the form:

$$M(z) = \begin{array}{c} cp \\ d_1 \\ \vdots \\ d_k \end{array} \begin{pmatrix} M_{11} & & & \\ M_{12} & M_{22} & & \\ \vdots & & \ddots & \\ M_{1n} & \ldots\ldots\ldots & & M_{nn} \end{pmatrix}$$

To prove that for all k, $\bar{C}_n^{d_k}$ is asymptotically polynomial we have to show that for all $i < \dim(M)$, $M_{ii}(\rho, \tau) = 1$.[4]

Proposition 1. *For all $1 \leqslant i \leqslant dim(M)$, $M_{ii}(\rho, \tau) = 1$.*

Proof. Let $r(\alpha)$ be the arity of the constructor symbol $\alpha \in \Omega$. From the first and second condition, we use relation (1) and get:

$$\begin{aligned} M_{ii}(z) &= \textstyle\sum_{\alpha \in \Omega} r(\alpha) z T(z)^{r(\alpha)-1} \\ &= z\Phi'(T(z)) \end{aligned} \tag{8}$$

Since $\rho = 1/\Phi'(\tau)$, $M_{ii}(\rho, \tau) = 1$ for all i. □

The next theorem will state that $\bar{C}_n^{d_k} = O(n^{k/2+1})$.

Theorem 5.4. *Let $\{d_k\}_k \cup \{cp\}$ be a set of derivative operators with d_k standing for the kth derivative and cp for the copy operator, such that:*

1. *For each k, the arity of d_k is 1.*
2. *For all $e \in E_{d_k}$, there is only one occurrence of $d_k(x_i)$ in D_e for each $x_i \in X(R_e^k)$.*
3. *There exists an occurrence of $d_{k-1}(x_i)$ for some $x_i \in X(R_e^k)$ for any integer $k > 1$ among the right members of the R_e^k's.*
4. *For all $e \in E_{d_k}$ there is no occurrence of d_i with $i > k$ in the right member D_e.*

then

$$\bar{C}_n^{d_k} \sim c_k n^{1+k/2}$$

Proof. The formal differentiation system structure is a chain structured one. Since every entry implicitly depends on $T(z)$, we know from our discussion above that operator d_i's order of growth inductively raises of $n^{1/2}$ on the way from d_i to d_{i+1} for all i.

Since operator copy has an average complexity of order n, $C_n^{d_k} = O(n^{1+k/2})$. □

Example 14. Let us consider a formal differentiation system with three operators:

- Derivative d_1
- Second derivative d_2
- the copy operator cp.

computing over terms built from constructors a (constant), exp (unary) and $*$ (binary).

Here are the rules of this system:

$$\begin{array}{llll}
R_1^{d_1} & d_1(1) & \longrightarrow & 1 \\
R_2^{d_1} & d_1(*(u,v)) & \longrightarrow & +(*(d_1(u), cp(v)), *(cp(u), d_1(v))) \\
R_3^{d_1} & d_1(exp(u)) & \longrightarrow & *(d_1(u), exp(cp(u))) \\
 & & & \\
R_1^{d_2} & d_2(1) & \longrightarrow & 0 \\
R_2^{d_2} & d_2(*(u,v)) & \longrightarrow & +(+(*(d_2(u), cp(v)), *(d_1(u), d_1(v))), +(*(d_1(u), d_1(v)), *(d_2(v), cp(u)))) \\
R_3^{d_1} & d_2(exp(u)) & \longrightarrow & +(*(d_2(u), exp(cp(u))), *(d_1(u), *(d_1(u), exp(cp(u))))) \\
 & & & \\
R_1^{cp} & cp(1) & \longrightarrow & 1 \\
R_2^{cp} & cp(*(u,v)) & \longrightarrow & *(cp(u), cp(v)) \\
R_3^{cp} & cp(exp(u)) & \longrightarrow & exp(cp(u))
\end{array}$$

[4] In this case, M is a lower triangular matrix, thus blocks have in fact one entry

We immediately deduce its representation matrix:

$$M = \begin{array}{c} cp \\ d_1 \\ d_2 \end{array} \begin{pmatrix} z+2zT(z) & 0 & 0 \\ z+2zT(z) & z+2zT(z) & 0 \\ z+2zT(z) & 2z+4zT(z) & z+2zT(z) \end{pmatrix}$$

We see this system obeys the conditions of theorem 5.4. Moreover, all the diagonal entries are equal to $z\Phi'(T(z)) = z+2zT(z)$ with $\Phi(u) = 1+u+u^2$. Graphically, the operators of this system are represented like this:

$$\begin{array}{ccccc} cp & & d_1 & & d_2 \\ & \longrightarrow & & \longrightarrow & \\ c_1 n & & c_2 n^{3/2} & & c_3 n^2 \end{array}$$

Actually, an effective computation of the c_i's gives:

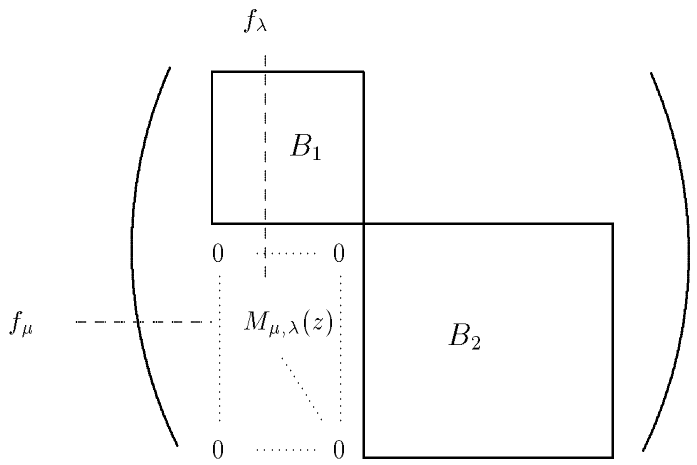
$$cp \sim n \qquad d_1 \sim 1/2\sqrt{3\pi} n^{3/2} \qquad d_2 \sim 3n^2 \qquad \blacklozenge$$

5.3.3. *Block-triangular matrices.* As shown in section 4.2, if a regular rewriting system is represented by an irreducible matrix then its operators average complexities have the same asymptotic behavior.

With an adapted definition of dependency between blocks of operators, the properties proved for triangular matrices hold for block-triangular matrices.

Definition 5.3. An irreducible block B_1 is said to depend on irreducible block B_2 if there exists an operator f of B_1 and an operator g of B_2 such that f depends on g.

We illustrate this definition treating an example, where the representation matrix has the following form:



In this figure, B_1 and B_2 are irreducible blocks and f_μ depends on f_λ. Then we apply Cramer's rule to block B_2 and $T = (T^{r_j}(z), \dots, T^{r_\mu}(z) + M_{\mu,\lambda}(z)C_{f_\lambda}(z), \dots, T^{r_n}(z))$, where $j = n+1-\dim(B_2)$:

$$C_{f_\mu}(z) = \frac{M_{\mu,\lambda}(z)B_2^*(\mu,\mu)}{\det(\mathrm{Id}-B_2)} C_{f_\lambda}(z) + \frac{\det^{[\mu]}(\mathrm{Id}-B_2)}{\det(\mathrm{Id}-B_2)}, \tag{9}$$

where $B_2^* = \mathrm{adj}(\mathrm{Id}-B_2)$. Using the same argument as in lemma 4 with $z = T(z)/\Phi(T(z))$, polynomials in $T(z)$: $M_{\mu,\lambda}(z)B_2^*(\mu,\mu)$ and $\det(\mathrm{Id}-B_2)$ are prime. Consequently, functions f_μ and f_λ behave as representant of their respective block and follow the hierarchy of the operators of a triangular representation matrix. This characteristic is independent of the number of functions in B_2 that depend on functions in B_1 as shows the previous formula.

Generalized to a representation matrix of the form:

$$\begin{pmatrix} B_1 & & & 0 \\ & B_2 & & \\ \vdots & \ddots & \ddots & \\ & \cdots & & B_n \end{pmatrix}$$

in which operator f_μ belongs to block B_n, formula 9 becomes:

$$C_\mu(z) = \sum_{\substack{\{B_i \mid B_n \text{ depends on } B_i\} \\ \{\lambda_i \mid f_\mu \text{ depends on } f_{\lambda_i}\}}} \frac{A_{\mu,\lambda_i}(z)}{\det(\mathrm{Id} - B_n)} C_{\lambda_i}(z) + \frac{\det^{[\mu]}(\mathrm{Id} - B_n)}{\det(\mathrm{Id} - B_n)}$$

where $A_{\mu,\lambda_i}(z) = M_{\mu,\lambda_i}(z) B_n^*(\mu,\mu)$.

Note that this expression is similar to the expansion of lemma 6 found for a one-operator block. The last term arises in both expansions: $\frac{\det^{[n,\mu]}(\mathrm{Id}-B_n)}{\det(\mathrm{Id}-B_n)}$ becomes $\frac{T^{r_n}(z)}{1-M_{f_\mu,f_\mu}}$ in a triangular matrix. This final term says that block B_n depends on itself as a connected component of operators.

Besides, it is another way to see that the irreducibility property is locally preserved in the diagonal blocks.

6. General case

This part aims at giving a systematic way to compute operators average complexities of any rewriting system.

6.1. About the exponential case.

The techniques we use are globally the same as in the polynomial case. Though the position of ρ_{0i} with respect to ρ induces small practical differences, it induces strong differences of complexities.

Theorem 6.1. *Let $\{f_i\}$ be a set of rewriting operators of a system represented by irreducible matrix M. Assume for each i, the dominant singularity of $C_{f_i}(z)$ is $\rho_{0_i} < \rho$.*

For all i, $\bar{C}_n^{f_i} \sim c_{f_i} \alpha^n n^{3/2}$, where $\alpha = \rho/\rho_{0_i}$.

Proof. We prove that the multiplicity of ρ_{0i} in $\det(\mathrm{Id}-M)$ is 1 using the same argument as in the proof of theorem 5.1.

The only difference is that there is no possibility for the multiplicity of factor $(T(z) - \tau_{0i})$ to exceed 1. It comes from the expansion of $T(z)$ around ρ_{0i}:

$$T(z) = \tau_{0i} - (1 - z/\rho_{0i}) \left(\frac{1}{\tau_{0i}} - \frac{\Phi'(\tau_{0i})}{\Phi(\tau_{0i})} \right)^{-1} + O((1 - z/\rho_{0i})^2)$$

Consequently, the expansion of $M_{ij}(z)$ and therefore of $\det(\mathrm{Id}-M)$ is always of the form: $g(z) + h(z)(1 - z/\rho_{0i})$.

Then, singularity analysis yields: $\bar{C}_n^{f_i} \sim c_{f_i} \alpha^n n^{3/2}$. □

6.2. Finding the dominant singularity.

This part focuses on spotting the cost series radii of convergence. Using the expansions of lemma 6 and its block equivalent, our intention is to translate them graphically by going down the diagram and collecting singularities information.

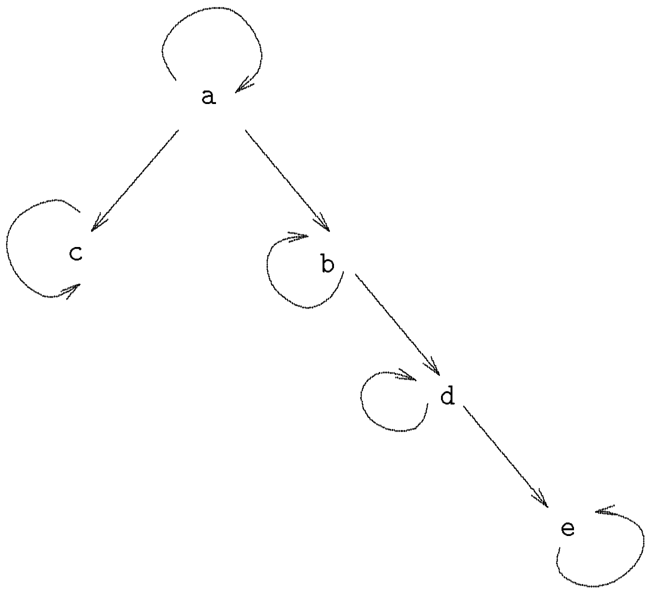
Once the representation matrix is turned into a triangular block matrix, you have to compute the blocks *proper dominant singularities* — i.e. the dominant singularities of each diagonal block, independently of the others — from the root block to the block we want to know the order of complexity.

Principle 1. The cost series dominant singularity of operators represented by an irreducible block B is the smallest proper singularity of all the blocks in the paths from the root blocks to B.

Example 15. Let a, b, c, d, e denote the diagonal irreducible blocks of the system represented by a matrix of the form:

$$\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \begin{pmatrix} \bigstar & & & & \\ \bigstar & \bigstar & & & \\ \bigstar & 0 & \bigstar & & \\ 0 & \bigstar & 0 & \bigstar & \\ 0 & 0 & 0 & \bigstar & \bigstar \end{pmatrix}$$

Its associated graph is as follows:



Suppose you analyze an operator of block e. For each block of the path going from a to e we compute ρ_k using the eigenvalue method for each corresponding block in M, and the minimal ρ_k is the dominant singularity of $C_{f_e}(z)$ for each operator f_e of block e. ♦

Example 16. We take again the example of the shuffle of trees. We introduce operator h and its rules are:

$$\begin{array}{llcl} R_1^h & h(a,a) & \longrightarrow & a \\ R_2^h & h(a, o(u,v)) & \longrightarrow & o(h(u,v), f(u,v)) \\ R_3^h & h(o(u,v), a) & \longrightarrow & o(h(u,v), g(u,v)) \\ R_4^h & h(o(u_1,v_1), o(u_2,v_2)) & \longrightarrow & o(h(u_1,u_2), h(v_1,v_2)) \end{array}$$

The representation matrix becomes:

$$\begin{array}{c} f \\ g \\ h \end{array} \begin{pmatrix} 2z^2T^2(z) & 2z^2 & 0 \\ z^2T^2(z) & 2z^2 + z^2T^2(z) & 0 \\ z^2T^2(z) + z^2 & z^2 & z^2T^2(z) + 2z^2 \end{pmatrix}$$

This matrix has a two irreducible square block diagonal: $A_1 = (M_{ij})$ for $i, j \in \{1, 2\}$ and $A_2 = M_{33}$ alone. We found in example 5 that the operators of A_1 had a linear average complexity; we deduce that $\rho_1 = \rho$.

Compute $r_3(\rho, \tau)$. We find that $r_3(\rho, \tau) < 1$; consequently ρ_1 is also the dominant singularity of $C_h(z)$. Thus, h has also a linear average complexity. ♦

6.3. **Polynomial growth of the average complexity.**

Definition 6.1. Let f be an operator of irreducible block B. The average complexity of f is computed recursively along all the paths going from the root blocks to B. We call the *main path(s)* of block B the path(s) going from the root block(s) to B after deleting all the paths that do not give the right order of the average complexity of f.

Assume the dominant singularity ρ_0 is encountered k times along the main path. If $\rho_0 = \rho$ then just apply the techniques seen in the preceding section. Otherwise, $\rho_0 < \rho$, then invariantly all the final block operators average complexities are of order $(\rho/\rho_0)^n n^{1+k/2}$

Principle 2. Let ρ_{0_i} be operator f_i's dominant singularity of block B.
Dependencies between blocks with proper singularities identical to ρ_{0_i} induce the polynomial growth of the $\bar{C}_n^{f_i}$'s asymptotic equivalent.

Namely, each block taken from the main path gives its contribution to $C_{f_i}(z)$ provided that their proper singularities are all ρ_{0_i}. Hence the results of theorem 3.2.

Example 17. Suppose we take the modified shuffle of example 16. We replace rule R_2^h by:

$$R_2^h \quad h(a, o(u,v)) \quad \longrightarrow \quad o(o(h(u,v), f(u,v)), h(u,v))$$

The representation matrix becomes:

$$\begin{array}{c} f \\ g \\ h \end{array} \begin{pmatrix} 2z^2T^2(z) & 2z^2 & 0 \\ z^2T^2(z) & 2z^2 + z^2T^2(z) & 0 \\ z^2T^2(z) + z^2 & z^2 & z^2T^2(z) + 3z^2 \end{pmatrix}$$

Every operator cost series in block A_1 has the same dominant singularity ρ therefore since all the entries of M depend on $T(z)$, the operators of block A_1 have a linear average complexity whereas $\bar{C}_n^h = O(n^{3/2})$ since $r_3(\rho, \tau) = 1$. ♦

Example 18. Now, we replace rule R_2^h and R_3^h by:

$$\begin{array}{lll} R_2^h \quad h(a, o(u,v)) & \longrightarrow & o(o(h(u,v), f(u,v)), h(u,v)) \\ R_3^h \quad h(o(u,v), a) & \longrightarrow & o(h(u,v), h(u,v)) \end{array}$$

The representation matrix becomes:

$$\begin{array}{c} f \\ g \\ h \end{array} \begin{pmatrix} 2z^2T^2(z) & 2z^2 & 0 \\ z^2T^2(z) & 2z^2 + z^2T^2(z) & 0 \\ z^2T^2(z) + z^2 & 0 & 4z^2 \end{pmatrix}$$

Operators f and g still have a linear order of growth while operator h has an average complexity of order n^2. Indeed, the dominant singularity of operator h cost series is still ρ but its diagonal entry in M does not depend on $T(z)$. ♦

Example 19. We change now rules R_2^h and R_3^h like this:

$$\begin{array}{lll} R_2^h \quad h(a, o(u,v)) & \longrightarrow & o(o(h(u,v), h(u,v)), h(u,v)) \\ R_3^h \quad h(o(u,v), a) & \longrightarrow & o(h(u,v), h(u,v)) \end{array}$$

The representation matrix becomes:

$$\begin{array}{c} f \\ g \\ h \end{array} \begin{pmatrix} 2z^2T^2(z) & 2z^2 & 0 \\ z^2T^2(z) & 2z^2 + z^2T^2(z) & 0 \\ z^2T^2(z) & 0 & 5z^2 \end{pmatrix}$$

Operators f and g have the same linear order of growth and this time operator h has an exponential order of growth.
Since $r_3(\rho, \tau) = 5/4 > 1$, we have $\rho_0 = 1/\sqrt{5} < \rho$ (we recall that $\rho = 1/2$). Consequently:

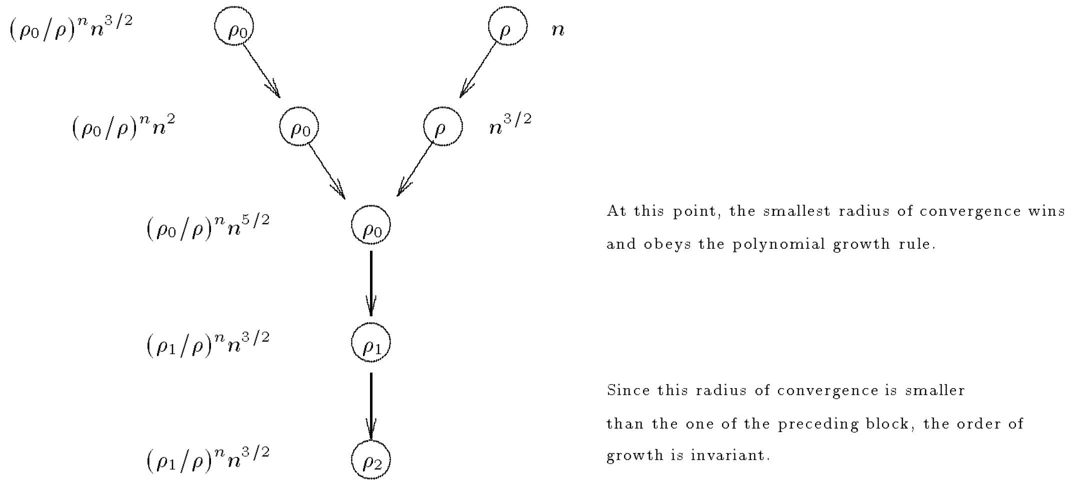
$$\bar{C}_n^h \sim c \left(\frac{\sqrt{5}}{2}\right)^n n^{3/2} \qquad ♦$$

6.4. **Synthesis: algorithmic approach.** We give here the main steps of a graphical determination of the order of growth of a given irreducible block B extracted from a general representation matrix. Basically, we start from the root blocks and we compute block proper singularities while storing the multiplicity of the current dominant singularity for each path to block B. More precisely:

- Identify the root blocks
- Make a traversal collecting information useful to establish the order of growth of an operator (single operator block) or a set of operator (irreducible diagonal block), that is:
  - Proper block dominant singularity.
  - If this block dominant singularity equals $T(z)$ dominant singularity then check dependency on $T(z)$.
- If the arity of a node is greater than 1 then apply the "strongest wins" rule to its children.

The next example will simulate the action of this algorithm on a system with a node of arity 2.

Example 20. Let $0 < \rho_1 < \rho_0 < \rho_2 < \rho$ where ρ is the term enumerative series $T(z)$ singularity. We suppose here that all polynomial blocks have entries that depend on $T(z)$. We put block proper singularities in the graph rather than operators to see the evolution of block operators orders of growth.



7. Conclusion: simulating rewriting systems

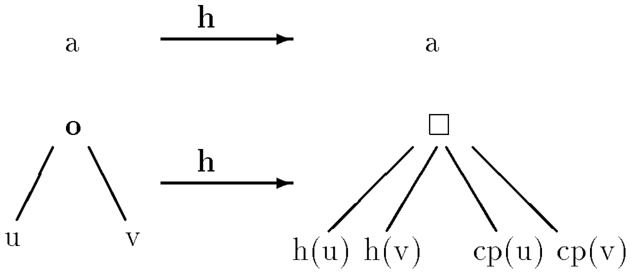
Various enumerative problems in combinatorics can be simulated with rewriting systems. We take the example of path lengths in binary trees .

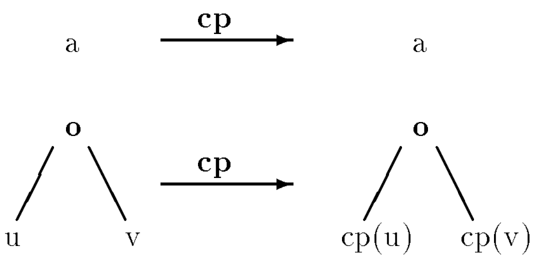
Example 21. We can simulate the computation of binary trees path length evaluating the cost series of a regular rewriting system operator.

Let $\mathcal{B} = o(\mathcal{B}_1, \mathcal{B}_2)$ be a binary tree. The path length of tree $\mathcal{B}$, $L(\mathcal{B})$, is recursively defined as follows:

$$L(\mathcal{B}) = L(\mathcal{B}_1) + L(\mathcal{B}_2) + |\mathcal{B}_1| + |\mathcal{B}_2|$$

We simulate it by the action of operator h over term trees built out of constructors a and o. cp denotes the copy operator. We obtain the following system:





Its related representation matrix is:

$$\begin{array}{c} cp \\ h \end{array} \begin{pmatrix} 2zT(z) & \\ 2zT(z) & 2zT(z) \end{pmatrix}$$

This triangular matrix behaves like the formal differentiation one (see example (14)) since its diagonal entries are equal to 1 in ρ, τ.

As expected, the copy operator has a linear order of growth and the average complexity of operator h is of order $n^{3/2}$. Combinatorially, $\bar{C}_n^h$ corresponds to the mean path length of a binary tree of size n. ⧫

The question is how far we can express such enumerative problems in terms of rewriting systems. We actually imagine there exists a class of combinatorial objects translated into a "rewriting system language" making their enumeration systematical.

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