How to compute minimal bases using Padé approximants

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Abstract

In this article, we present a new method to compute minimal bases for the kernel of a polynomial matrix. This method uses the computation of Padé approximants for the special case where one does not look after an approximation but an exact value. For a polynomial matrix of order m and degree d, its cost is about $\mathcal{O}(m^4d^2)$ which may be better of the cost of a previous method by Beleen and Van Dooren [4].

1 Introduction

A minimal base for a submodule $\mathcal{M}$ of $K^p[z]$ is a base which components have minimal degrees [5]. The computation of a minimal base for the kernel of a polynomial matrix $M(z)$ may be useful to establish the singular part of a matrix pencil (when $M(z)$ is a matrix pencil) [6], or to find some special form for polynomial matrix like column reduced form [7].

A previous method to compute such a base, was introduced by Beelen and Van Dooren [4]. To compute a minimal bases for the Kernel of a polynomial matrix $M(z)$ of order m and degree d, they turn the problem into a pencil problem of order md. Then, they use classical algorithms to deal with matrix pencil [11, 10, 3]. If the complexity of these last algorithms is quite good, this method lost efficiency by considering that a matrix problem of order m and degree d is a pencil problem of order md.

Our method has a completly different approach. It is based on the computation of Padé approximants for the matrix $M(z)$, *i.e.* a base for the solutions of the equation

$$M(z)p(z) = \mathcal{O}(z^N) \tag{1}$$

where $p(z)$ is a polynomial vector and N is called the *approximation order*. Using the fact that $M(z)$ is not a matrix which components are formal series like in problem of Padé approximants, but a polynomial matrix, we know that it exists an approximation order N such that a base for the solution of (1) describe the solutions of minimal degrees of

$$M(z)p(z) = 0.$$

Moreover, recents methods, from Beckermann and Labahn [1] and Van Barel and Bultheel [9], to compute vector Padé approximant, give the solution in the form of special bases which look like minimal bases. Using these methods specialized to our case, we find a cost about $\mathcal{O}(m^4d^2)$ arithmetic operations.

In the section 2 of this paper we give the definition and some properties of a minimal base. The section 3 is devoted to the description of the vector Hermite-Padé problem and to its solutions given by [8, 9, 1]. These solutions are given into the form of s-reduced bases or σ-bases. We show that this kind of bases are generalizations for the notion of minimal bases. Then, in section 4, we give results which allow to use the algorithm of [1] to compute a minimal base for the kernel of a polynomial matrix. Section 5 give a simplified description of the algorithm of [1] specialized to our case. We give a specialization of the cost for this algorithm in section 6.

2 Minimal bases

In this section we recall the definition and some properties of a minimal base. We define the column reduced form for a polynomial matrix and give the equivalence with the notion of minimal base.

Definition 1 *Given $\mathcal{B} = (\mathbf{b}_1(z), ..., \mathbf{b}_m(z))$ a base for a $K^p[z]$-submodule $\mathcal{M}$. The base $\mathcal{B}$ is minimal if*

$$\sum_{i=1}^{m} deg(\mathbf{b}_i)$$

is minimal over the bases of $\mathcal{M}$.

Minimal bases for a submodule $\mathcal{M}$ are not uniquely defined, but all of them have the same set of degrees $\{deg(\mathbf{b}_i), 1 \leq i \leq m\}$. These degrees are sometime called the *Forney dynamical indices* [5].

We will often describe a base

$$\mathcal{B} = (\mathbf{b}_1(z), ..., \mathbf{b}_m(z))$$

using the matrice

$$B(z) = \left[\begin{array}{ccc} \mathbf{b}_1(z) & ... & \mathbf{b}_m(z) \end{array} \right]$$

Definition 2 *Given a (n, m) polynomial matrix $P(z)$. The degrees $d_i, 1 \leq i \leq m$ of the ith column of $P(z)$ are called the column degrees of $P(z)$. Let P_{hc} be the highest column degree coefficient matrix, a matrix whose ith column comprise the coefficients of z^{d_i} in the ith column of $P(z)$. The matrix $P(z)$ is said to be column reduced if $rank(P(z)) = rank(P_{hc})$.*

Property 1 *The column of a (p, m) polynomial matrix $B(z)$ describe a minimal base iff $rank(B(z)) = m$ and the matrix $B(z)$ is column reduced.*

3 Padé approximants

The vector Hermite-Padé problem is described in [1] as

Problem 1 *Let m, p, and N be integers, m, $p \geq 2$. Let $F_1(z)$, $F_2(z)$, ..., $F_m(z)$ be in $K^p[[z]]$, and let $\mathbf{n}= (n_1, ..., n_m)$ be a multiindex. Find linearly independant polynomial tuples $(p_1(z), ..., p_m(z))$, $p_i(z) \in K[z]$, such that $deg(p_i(z)) \leq n_i, 1 \leq i \leq m$ and*

$$F_1(z)p_1(z) + ... + F_m(z)p_m(z) = z^N R(z) \quad (2)$$

where $R(z) \in K^p[[z]]$.

Remark : If $F(z) = [F_1(z) \; ... \; F_m(z)]$ is a polynomial matrix (and not a matrix of formal power series) and if $R(z) = 0$, then $[p_1(z) \; ... \; p_m(z)]^t$ is an element of $\mathrm{Ker}(F(z))$. This is the way we will use solution of the vector Hermite-Padé approximant problem to compute a minimal base for the kernel of a polynomial matrix in the next section.

The solution to this problem is given into the form of *s-reduced base* [9] or *σ-base*, $\sigma = N$, [1] for the $K^p[z]$ submodule defined by the equation (2). In [7] it is shown that the two notions (s-reduced base and σ-base) are equivalent in the case of vector Hermite-Padé problem. Although the components of such a base do not satisfy the degree conditions, they allow to describe all the solutions of the vector Hermite-Padé problem [9, 1].

We will describe now the notion of s-reduced base, then it will be obvious that it is a generalization of the notion of minimal base.

Definition 3 ([9]) *Given a multiindex $\mathbf{s}= (s_1, ..., s_m)$, the $\mathbf{s}$-degree of a polynomial tuple $\mathbf{p}(z) = (p_1(z), ..., p_m(z))$ is defined as*

$$\mathbf{s}\text{-}deg(\mathbf{p}) = max\{deg(p_i) + s_i\}$$

The $\mathbf{s}$-highest degree coefficient of a polynomial tuple $(p_1(z), ..., p_m(z))$ with $\mathbf{s}$-degree δ is the m-tuple whose ith component is the coefficient of $z^{\delta - s_i}$ in $p_i(z)$.

A set of polynomial m-tuple is $\mathbf{s}$-reduced iff the $\mathbf{s}$-highest degree coefficients are linearly independent.

As we said before, this definition is a generalization of the notion of minimal base. As a matter of fact, if $\mathbf{s}= (0, 0, ..., 0)$, the $\mathbf{s}$-degree of a polynomial tuple is its usual degree.

Example 1 *Given a multiindex $\mathbf{s}$ and the matrix*

$$P(z) = \left[\begin{array}{cc} \mathbf{p}_1 & \mathbf{p}_2 \end{array} \right] = \left[\begin{array}{cc} 2z & z^2 \\ 1 & 2z \end{array} \right]$$

we want to know if the columns of $P(z)$ are $\mathbf{s}$-reduced. Assume that $\mathbf{s}= (1, 2)$, then

$$\mathbf{s}\text{-}deg(\mathbf{p}_1) = max\{1 + 1, 0 + 2\} = 2$$
$$\mathbf{s}\text{-}deg(\mathbf{p}_2) = max\{2 + 1, 1 + 2\} = 3$$

and the matrix whose columns are the $\mathbf{s}$*-highest degree coefficients of the columns of* $P(z)$ *is given by*

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

So the columns of $P(z)$ *are* $\mathbf{s}$*-reduced. Assume now that* $\mathbf{s}= (0,0)$*, then*

$$\mathbf{s}\text{-}deg(\mathbf{p}_1) = deg(\mathbf{p}_1) = 1$$
$$\mathbf{s}\text{-}deg(\mathbf{p}_2) = deg(\mathbf{p}_2) = 2$$

and the matrix whose columns are the $\mathbf{s}$*-highest degree coefficients of the columns of* $P(z)$ *is* P_{hc} *(see definition 2)*

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

The columns of $P(z)$ *are not* $\mathbf{s}$*-reduced, or, equivalently,* $P(z)$ *is not column reduced.*

In [9, 1], the solution to the vector Hermite-Padé problem of order N is given by a $\mathbf{s}$-reduced base of the submodule $\mathcal{S}_N$ defined by the equation (2). The components of such a base do not verify the problem because they do not verify the degree conditions. Nevertheless, they allow to parametrize all solutions of the problem [9, 1], this property is a consequence of the next theorem.

Theorem 1 ([9]) *Given an* $\mathbf{s}$*-reduced basis for a submodule* $\mathcal{S}$ *with basis elements* $\mathbf{b}_i$ *having* $\mathbf{s}$*-degree* δ_i*, all the elements of the submodule* $\mathcal{S}$ *having* $\mathbf{s}$*-degree* $\leq \delta$ *can be parametrized uniquely as*

$$\sum_{i=1}^{m} c_i \mathbf{b}_i$$

with c_i *a polynomial of degree* $\leq \delta - \delta_i$*.*

In this paper, we are not interested in this parametrization but in the property of degrees (or $\mathbf{s}$-degrees) minimality along an $\mathbf{s}$-reduced base for $\mathcal{S}_N$. As a matter of fact, $\mathcal{S}_N$ contains the solutions of

$$F_1(z)p_1(z) + ... + F_m(z)p_m(z) = 0 \qquad (3)$$

In the next section, we will show that if $F(z)$ is a polynomial matrix, for some N and for $\mathbf{s}= (0,0,...,0)$, an $\mathbf{s}$-reduced base of $\mathcal{S}_N$ contains a minimal base for the submodule defined by the equation (3).

4 Computation of minimal bases

Given a polynomial matrix $F(z)$, we want to compute a minimal base for its kernel, or, which is the same, to find a column reduced matrix $P(z)$, of maximal rank, such that

$$F(z)P(z) = 0$$

The next theorem allows to compute this base using computation of an $\mathbf{s}$-reduced base for the vector Hermite-Padé approximation of $F(z)$.

Theorem 2 *Let* $\mathbf{s}= (0,0,...,0)$ *and* $F(z)$ *be a* (p,m) *polynomial matrix of degree* d*. Let* $\mathcal{B}^N$ *be a* $\mathbf{s}$*-reduced base for the vector Hermite-Padé problem of* $F(z)$ *of order* N*. If*

$$N \geq max\{d_k, 1 \leq k \leq l\} + d + 1$$

where l *is the dimension of* $ker(F(z))$ *and the* d_k *are the degrees of the vectors of a minimal base for* $ker(F(z))$*, then* $\mathcal{B}^N$ *contains a minimal base of* $ker(F(z))$*.*

Proof : Assume that d_1 is the minimal degree along the vectors of a minimal base of $\ker(F(z))$. We will proove first that $\mathcal{B}^N$ contains a vector $\mathbf{b}$ of degree d_1 such that $F(z)\mathbf{b}(z) = 0$. Let $\mathbf{p}_1$ be a vector of $\ker(F(z))$ of degree d_1. As $\mathbf{s}= (0,0,...,0)$, we have

$$\mathbf{s}\text{-}deg(\mathbf{p}_1) = deg(\mathbf{p}_1) = d_1$$

We note $\mathbf{b}_1$, $\mathbf{b}_2$, ..., $\mathbf{b}_m$, the vectors of $\mathcal{B}^N$ and δ_1, δ_2, ..., δ_m, their degrees. Using theorem 1, we know that there exist polynomials c_1, c_2, ..., c_m such that

$$deg(c_i) \leq d_1 - \delta_i \ , \ \ 1 \leq i \leq m \text{ and}$$
$$\mathbf{p}_1 = \sum_{i=1}^{m} c_i \mathbf{b}_i$$

As $\mathbf{p}_1 \neq 0$, it exist h such that $deg(c_h) \geq 0$ that is $d_1 \geq \delta_h$. While $\mathbf{b}_h$ is a vector of $\mathcal{B}^N$, it verifies

$$F_1(z)b_{h1}(z) + ... + F_m(z)b_{hm}(z) = z^N R(z)$$

But on the left hand side of the equation we have a polynomial of maximal degree $d_1 +$

$deg(F(z)) = d_1 + d$ and, by assumption, $N \geq d_1 + d + 1$. So

$$F_1(z)b_{h1}(z) + ... + F_m(z)b_{hm}(z) = 0$$

and $\delta_h = d_1$ because d_1 is minimal.

Now, assume that $\mathcal{B}^N$ contains $k-1$ vectors of a minimal base of $\ker(F(z))$ of minimal degrees d_1, d_2, ..., d_{k-1}. We note $\mathbf{b}_1$, $\mathbf{b}_2$, ..., $\mathbf{b}_{k-1}$ these vectors. Let $\mathbf{p}_k$ be a vector of minimal degree d_k, linearly independant from the $\mathbf{b}_i$, $1 \leq i \leq k-1$, which verifies $F(z)\mathbf{p}_k = 0$. We have to show that it exist a vector $\mathbf{b}$ in $\mathcal{B}^N$, of degree d_k, which verifies $F(z)\mathbf{b} = 0$. There exist polynomials $\tilde{c}_1$, $\tilde{c}_2$, ..., $\tilde{c}_m$ such that

$$deg(\tilde{c}_i) \leq d_k - \delta_i \ , \quad 1 \leq i \leq m \text{ and}$$
$$\mathbf{p}_k = \sum_{i=1}^{m} \tilde{c}_i \mathbf{b}_i$$

As $\mathbf{p}_k$ is linearly independant from the $\mathbf{b}_i$, $1 \leq i \leq k-1$, there exists at least one element $\mathbf{b}_{\tilde{h}}$ in $\mathcal{B}^N$ such that $\tilde{h} > k-1$ and $d_k \geq \delta_{\tilde{h}}$. While $\mathbf{b}_{\tilde{h}}$ is a vector of $\mathcal{B}^N$, it verifies

$$F_1(z)b_{\tilde{h}1}(z) + ... + F_m(z)b_{\tilde{h}m}(z) = z^N R(z)$$

But on the left hand side of the equation we have a polynomial of maximal degree $d_k + d$ and, by assumption, $N \geq d_k + d + 1$. So

$$F_1(z)b_{\tilde{h}1}(z) + ... + F_m(z)b_{\tilde{h}m}(z) = 0$$

and $\delta_{\tilde{h}} = d_k$ because d_k is minimal. □

The next theorem allows to give an upper bound for N which depend only on the initial datas.

Theorem 3 *Let $F(z)$ be a (p, m) polynomial matrix of degree d. Assume that a minimal base of $ker(F(z))$ possess l vectors of degrees d_k, $1 \leq k \leq l$, then*

$$\sum_{k=1}^{l} d_k \leq (m-1)d$$

Proof : If $F(z))$ is a matrix pencil ($d = 1$), the d_k, $1 \leq k \leq l$ are the right kronecker indices of $F(z)$ [6]. Then, using the normal form of $F(z)$ we can easily proove the theorem.

Now assume that $F(z) = F_0 + F_1 z + ... + F_d z^d$. Consider the linearised pencil defined by :

$$z \begin{bmatrix} Id_m & & & \\ & \ddots & & \\ & & Id_m & \\ & & & F_d \end{bmatrix} + \begin{bmatrix} -Id_m & & & \\ & \ddots & & \\ & & -Id_m & \\ F_0 & \dots & \dots & F_{d-1} \end{bmatrix}$$

The sum of the degrees of the vectors of a minimal base for the kernel of this pencil is less than $md-1$. Moreover, $(\mathbf{b}_1, ..., \mathbf{b}_l)$ is a minimal base for $F(z)$ iff the columns of

$$\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_l \\ z\mathbf{b}_1 & z\mathbf{b}_2 & \dots & z\mathbf{b}_l \\ & & & \\ z^{d-1}\mathbf{b}_1 & z^{d-1}\mathbf{b}_2 & \dots & z^{d-1}\mathbf{b}_l \end{bmatrix}$$

form a minimal base for the linearised pencil. We have

$$\sum_{i=1}^{l} (deg(\mathbf{b}_i + (d-1)) \leq md - 1$$

thus, if we assume that $l > 0$, that is $\ker(F(z)) \neq \emptyset$,

$$\begin{aligned} \sum_{i=1}^{l} (deg(\mathbf{b}_i)) &\leq md - 1 - l(d-1) \\ &\leq (m-1)d \end{aligned}$$

□

Using the algorithms given by [9, 1] to compute solutions of the vector Hermite-Padé problem for a (p, m) polynomial matrix $F(z)$ of degree d, for an order $\geq md + 1$, it is possible to obtain a minimal base for $\ker(F(z))$. In the next section, we present a specialization of an algorithm by [1].

5 Algorithm

The algorithm described here is a matriciel version of the algorithm given by [1] to compute σ-base for a matrix $F(z)$ of formal series. As we have said before, in our context, σ-base is the same notion as $\mathbf{s}$-reduced base. Moreover we specify this algorithm to our particular case of computation of minimal base.

This is an iterative algorithm, it start with the base given by the columns of the identity matrix, thus the result is of order 0. Then, at the step i, it transform the base of order i to another one of order $i+1$.

The input of this algorithm is a polynomial matrix $F(z)$, the matrix $P^{(i)}(z) = [\mathbf{p}_1^{(i)} \ldots \mathbf{p}_m^{(i)}]$ denotes a $\mathbf{s}$-reduced base of order i for $F(z)$ (where $\mathbf{s} = (0, ..., 0)$). The integers $d_k^{(i)}$ for $k = 1, ..., m$ denote the respective degrees of the $\mathbf{p}_k^{(i)}$. The matrix $C^{(i)}$ is the constant matrix such that

$$F(z)P^{(i)}(z) = C^{(i)}z^i + \mathcal{O}(z^{i+1}).$$

Algorithm

Input: $F(z)$, a (p, m) polynomial matrix.
N, the order.
Initializations:
$P^{(0)}(z) = [\mathbf{p}_1^{(0)} \ldots \mathbf{p}_m^{(0)}] = Id_m$
$d_k^{(0)} = 0$ for $k = 1, ..., m$
Iterative step:
For $i = 0, 1, ..., N$ do
For $r = 1, 2, ...p$ do
$z^{-i}F(0)P^{(i)}(0) = C^{(i)} = [c_{kl}^{(i)}]$
$\Lambda_i = \{l \ : \ c_{r,l} \neq 0\}$
If $\Lambda_i \neq \{\}$ Then
Let $\pi \in \Lambda_i$ such that
$d_\pi^{(i)} = min\{d_l^{(i)}, l \in \Lambda_i\}$.
(* elimination *)
For $k = 1$ to m, $k \neq \pi$ do
$\mathbf{p}_k = \mathbf{p}_k - \frac{c_{i,k}}{c_{i,\pi}}\mathbf{p}_\pi$
EndDo
(* multiplication *)
$\mathbf{p}_\pi = z\mathbf{p}_\pi$
$d_\pi^{(i)} = d_\pi^{(i)} + 1$
EndIf
EndDo
For $k = 1, ..., m$ do
$\mathbf{p}_k^{(i+1)} = \mathbf{p}_k^{(i)}$
$P^{(i+1)} = [\mathbf{p}_1^{(i+1)} \ldots \mathbf{p}_m^{(i+1)}]$
$d_k^{(i+1)} = d_k^{(i)}$
EndDo
EndDo
Output: $P^{(N)}(z)$ an order N $\mathbf{s}$-reduced base for $F(z)$, where $\mathbf{s} = (0, ..., 0)$.

According to theorems 2 and 3, we know that if $N \geq md + 1$, the base $P^{(N)}(z)$ contains a minimal base for $ker(F(z))$.

The cost given in [1] does not consider the particularity of the problem when $\mathbf{s} = (0, ..., 0)$ and $F(z)$ is a polynomial matrix. So we will give a specialization of this cost in our case in the next section. On the other hand, in [1], the authors use a kind of linearization of the problem (and then of the algorithm), and present a fast algorithm based on recursivness and FFT fast polynomial multiplications. We have not find yet a way to specialize this last algorithm but we hope that it could reduce the cost given in the next section.

6 Cost

Theorem 4 *Given a (p, m) polynomial matrix of degree d, $F(z)$, the computation of a minimal base for $ker(F(z))$, using the Padé approximant method has a complexity of $\mathcal{O}(m^2p^2d^2)$ arithmetic operations ($\mathcal{O}(m^4d^2)$ if $p \sim m$).*

To proove this theorem we will use the

Lemma 1 *For the step i of the algorithm, the residual $F(z)P^{(i)}(z)$ is into the form:*

$$C^{(i)}z^i + C_{i+1}z^{i+1} + \ldots + C_{i+d}z^{i+d}.$$

Proof (lemma 1) **:** The property is true for $i = 0$. Assume that it is true for $i-1$, we will proove it for i.

For a given r, the column $\mathbf{p}_\pi$ of $P^{(i-1)}$ is called the pivot. When a column is a pivot, it is multiplied by z and its degree increase, then (at the step i) a column $\mathbf{p}$ of $P^{(i-1)}$ may be the pivot for one $r \in [1, p]$ only. Moreover, when a column of $P^{(i-1)}$ is a pivot, it will not take place anymore in the operation called "elimination" in the algorithm (during the step i). Finally, an operation of elimination between a pivot (whose residual is of degree $i-1+d$) and another column of $P^{(i-1)}$ of degree $i-1+d$ does not change the degree of these last column. So, for a given r:

- When a column $\mathbf{p}$ is a pivot, the residual for this column is changed from :

$$c_{i-1}z^{i-1} + \ldots + c_{i-1+d}z^{i-1+d}$$

to

$$c_{i-1}z^i + \ldots + c_{i-1+d}z^{i+d}$$

and then does not change anymore during the step i.

- When a column $\mathbf{p}$ is not a pivot, the residual for this column keep the form

$$c_{i-1}z^{i-1} + \ldots + c_{i-1+d}z^{i-1+d}$$

but at the end of the step i we know that $c_{i-1} = [0 \ldots 0]^t$.

$\square$

Proof (theorem 4) **:** To proove the theorem we will show that for a vector of the base, there were only $p(d+1)$ eliminations. As such elimination costs at most $((m-1)d)p$ arithmetic operations, the total cost will be about $\mathcal{O}(m^2p^2d^2)$ arithmetic operations.

In fact, the degree of the residual for a column of $P^{(i)}$ is bounded by $i+d-(n_{el} \bmod p)$ where n_{el} is the number of eliminations on the considered column from the step 0 to the step i of the algorithm. This is true for $n_{el}=0$ (see lemma 1). Assume that $n_{el}=0$ and that the residual of the considered column at the step i is on the form

$$c_iz^i + \ldots + c_{i+d}z^{i+d}$$

We call principal index γ, the index of the first component of c_i which is not equal to zero. During the step $i+1$ of the algorithm, for $r=1$ to $r=\gamma-1$, it does not make an alteration for the considered column. Then, when $r=\gamma$ there are two possibilities:

- the considered column is the pivot, so γ does not change and $n_{el}=0$ until the end of the step $i+1$.

- the considered column is not the pivot, so there is an elimination on it, $n_{el}=1$ and γ increase exept if all the components of c_i turn out to 0: in this case, the degree of the residual, at the end of the step $i+1$, is not $i+1+d$ but $i+d$.

So, in the worst case for a column, at each elimination on it, γ increase to $\gamma+1$ until $n_{el}+1 = \gamma = p$. Then, at the next elimination, the degree of the residual will be $n_{el} \bmod p = 1$ less than the "theorical" one (see lemma 1) for the considered step. This will be exactly the same for $n_{el} \bmod p \geq 1$.

Finaly, if $n_{el} = p(d+1)$ for a column of the base, the degree of the residual for this column is bounded by $i-1$ at the step i so it is equal to 0 and the column will not have elimination anymore. $\square$

7 Conclusion

In this paper, we give a new method to compute a minimal base for the kernel of a polynomial matrix. Given a polynomial matrix of dimension m and degree d, this new method costs about $\mathcal{O}(m^4d^2)$ aritmetic operation when the method given in [4] uses $\mathcal{O}(m^3d^3)+\mathcal{O}(lm^3d^2)$ arithmetic operations where l can be as big as md. In fact, as we said before, the method by [4] is better when the input matrix is a matrix pencil ($d=1$), but its efficiency decreases when d increases because of to the linearization of the input matrix.

Moreover, to compute the cost of this new method, we do not take all the advantages of the results from [1] about Padé approximants computation. So we hope that our theoretical cost can decrease.

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